Needed Reductions with Context-Sensitive Rewriting*

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Abstract. Computing with functional programs involves reduction of terms to normal form. When considering non-terminating programs, this is achieved by using some special, *normalizing* strategy which obtains the normal form whenever it exists. *Context-sensitive rewriting* can improve termination and also avoid useless reductions by imposing fixed, syntactic restrictions on the replacements. In this paper, we analyze the efficiency of context-sensitive computations with respect to the notion of *needed* reduction. As context-sensitive rewriting is complete in performing reductions to a root-stable form, we base our investigation on Middeldorp's theory of root-necessary reductions which is a generalization of Huet and Lévy's theory of (sequential) needed reductions to reductions leading to root-stable form both in sequential and parallel executions.

Keywords: functional programming, needed reductions, replacement restrictions, strategies, term rewriting systems.

1 Introduction

In *context-sensitive rewriting* (*csr*) [8], the evaluation of the arguments of a function call $f(t_1, \ldots, t_k)$ is restricted to some fixed argument positions of f which we denote as $\mu(f)$, and is forbidden for the others. The mapping $\mu : \Sigma \to \mathcal{P}(\mathbb{N})$ is called the *replacement map*. For example, by defining $\mu(\texttt{if}) = \{1\}$, we avoid reductions in second and third arguments of an input expression `if(and(true,false),s(0)+s(0),0+s(0))`. They do not make sense until the condition `and(true,false)` has been evaluated. The restrictions given by μ on arguments of function calls naturally extend to occurrences of terms. By imposing such replacement restrictions, we can also lose useful steps. Given a Term Rewriting System (TRS), $\mathcal{R}$, we have shown how to automatically compute the *canonical* replacement map $\mu^{com}_{\mathcal{R}}$ for $\mathcal{R}$. The replacement map $\mu^{com}_{\mathcal{R}}$ is proven to be the most restrictive one which is still able to achieve completeness in reductions leading to a root-stable form (i.e., a term which cannot be reduced to a

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redex) of the input term. We have shown how to fully evaluate a term (obtaining a constructor term or value) by using a less restrictive replacement map $\mu_{\mathcal{R}}^{\mathcal{B}}$ [8].

In this paper, we are interested in how to effectively perform such computations, rather than in just ensuring the *possibility* of having them. Thus, we want to define (normalizing) context-sensitive strategies which use the replacement restrictions efficiently. To formalize 'efficiency', we consider a suitable notion of neededness of the context-sensitive computations.

When considering orthogonal TRSs, Huet and Lévy's theory of *needed redexes* gives us a precise definition of what a needed reduction step is [4]. Middeldorp argues that reduction to root-stable form, and the corresponding notion of *root-needed redex* is even a more fundamental notion than other neededness notions, since root-needed normalization provides a unified framework to analyze reductions leading to a finite normal form (i.e., it includes Huet and Lévy's theory) and also reductions leading to an infinite normal form for which Huet and Lévy's theory does not apply [12]. He also generalizes Sekar and Ramakrishnan's work on necessary sets of redexes [14] from constructor-based TRSs to root-necessary sets of redexes in general TRSs. Thus the notion of root-necessary reduction, which applies in parallel strategies, is also available.

In this work, we prove that, given an orthogonal TRS $\mathcal{R}$, every non-root-stable term t has a *replacing* root-needed redex when considering the replacement map $\mu_{\mathcal{R}}^{com}$. Hence, by imposing replacement restrictions, we still make root-needed computations possible. We also show how to find the replacing root-needed redexes of a non-root-stable term. This is the basis for the definition of optimal root-normalizing *cs*-strategies and hence optimal evaluation *cs*-strategies.

We define the concept of μ-restricted strategy $\mathbb{F}^{\mu}$ for a given TRS, replacement map μ and rewriting strategy $\mathbb{F}$. We analyze the root-normalization properties of the μ-restriction of the usual strategies (parallel outermost, leftmost-outermost, etc.). We show that they are usually improved in some aspect. We also define two classes of TRSs for which we prove that the restricted strategies have normalizing properties, while the unrestricted ones do not. We define the *strongly replacing-independent* TRSs for which *every* *cs*-strategy is root-normalizing. We define the *μ-left-normal* TRSs, a superset of left-normal TRSs [13] for which the μ-restricted leftmost-outermost strategy is also root-normalizing. We also extend these results to perform finite, efficient normalization and evaluation.

In Section 2, we give some preliminary definitions. Section 3 recalls the basic concepts of *csr* and characterizes the *strongly replacing-independent* and *μ-left-normal* TRSs. Section 4 compares context-sensitive computations and root-needed reductions. Section 5 defines the concept of restricted strategy and illustrates the definition of root-normalizing and normalizing *cs*-strategies. Section 6 summarizes conclusions and outlines to future work.

2 Preliminaries

Let us first introduce the main notations used in the paper. For full definitions we refer to [2, 6]. Denote a countable set of variables as V. By Σ we denote a signature: a set of function symbols $\{\mathtt{f}, \mathtt{g}, \ldots\}$, each of which has a fixed arity given by a function $ar : \Sigma \to \mathbb{N}$. By $\mathcal{T}(\Sigma, V)$ we denote the set of terms. A k-tuple $t_1, \ldots, t_k$ of terms is denoted as $\tilde{t}$, where k will be clarified from the context. Given a term t, $Var(t)$ is the set of variable symbols in t.

Terms are viewed as labelled trees in the usual way. Occurrences $u, v, \ldots$ are represented by chains of positive natural numbers used to address subterms of t. Denote as $O(t)$ the set of occurrences of a term t. If u is an occurrence, and W is a set of occurrences, $u.W$ denotes the set of occurrences $\{u.v \mid v \in W\}$. Occurrences are ordered by the standard prefix ordering: $u \leq v$ iff $\exists w$ such that $v = u.w$. We use the total ordering $\leq_L$: $\epsilon \leq_L u$; if $u = i.u'$, $v = j.v'$, $u \leq_L v \Leftrightarrow i < j \vee (i = j \wedge u' \leq_L v')$. A linear term is a term having no multiple occurrences of the same variable. The subterm at occurrence u of t is $t|_u$. $t[s]_u$ is the term t with the subterm at the occurrence u replaced with s. The symbol labelling the root of t is $root(t)$. The set of symbols from Σ appearing in t is $\Sigma(t) = \{f \in \Sigma \mid \exists u \in O(t).\ root(t|_u) = f\}$. Given $T \subseteq \mathcal{T}(\Sigma, V)$, we define $\Sigma(T) = \cup_{t \in T} \Sigma(t)$. The set of nonvariable occurrences in a term t is $O_\Sigma(t) = \{u \in O(t) \mid root(t|_u) \in \Sigma\}$. The variable occurrences are $O_V(t) = O(t) \backslash O_\Sigma(t)$.

We refer to any term C, which is the same as t everywhere except below u, i.e. $\exists s$ such that $C[s]_u = t$, as the *context* within the replacement occurs. Roughly speaking, a context is a term C with a 'hole' at a specific occurrence u.

A rewrite rule is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\Sigma, V)$, $l \notin V$ and $Var(r) \subseteq Var(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A TRS is a pair $\mathcal{R} = (\Sigma, R)$, where R is a set of rewrite rules. The set of lhs's of $\mathcal{R}$ is $L(\mathcal{R})$. An instance $\sigma(l)$ of a lhs $l \in L(\mathcal{R})$ is a redex. Given a term t, denote as $O_\mathcal{R}(t)$ the set of redex occurrences in t, $O_\mathcal{R}(t) = \{u \in O(t) \mid \exists l \in L(\mathcal{R}).\ t|_u = \sigma(l)\}$. A TRS $\mathcal{R}$ is *left linear*, if $\forall l \in L(\mathcal{R})$, l is a linear term. Two rules $l \to r$ and $l' \to r'$ *overlap*, if there is a non-variable occurrence $u \in O_\Sigma(l)$ and a most general unifier σ such that $\sigma(l|_u) = \sigma(l')$. In this case, the pair $\langle \sigma(l)[\sigma(r')]_u, \sigma(r) \rangle$ is called a critical pair. A critical pair $\langle \sigma(l)[\sigma(r')]_u, \sigma(r) \rangle$ with $u = \epsilon$ is an overlay. A critical pair $\langle t, s \rangle$ is trivial if $t = s$. A TRS is almost non-ambiguous if all its critical pairs are trivial overlays. A TRS is non-ambiguous, if there are no overlapping lhs's (trivial overlap in the same lhs is not considered). A left-linear, almost non-ambiguous TRS is called almost orthogonal. A left-linear, non-ambiguous TRS is called orthogonal.

Given a TRS $\mathcal{R} = (\Sigma, R)$, a term t rewrites to a term s (at the occurrence u), written $t \stackrel{u}{\to}_\mathcal{R} s$ (or just $t \to s$), if $t|_u = \sigma(l)$ and $s = t[\sigma(r)]_u$, for some rule $l \to r \in R$, $u \in O(t)$ and substitution σ. A term t is root-stable (also said head-normal form), if there is no derivation $t = t_1 \to t_2 \to \cdots$ which reduces the root of a term t_i, $i \geq 1$. A term t is a normal form if $O_\mathcal{R}(t) = \emptyset$. A term t is weakly normalizing if there exists a normal form $\bar{t}$ such that $t \to^* \bar{t}$. A term t is strongly normalizing if there exists no infinite rewrite sequences $t = t_1 \to t_2 \to \cdots$. A TRS $\mathcal{R}$ is weakly normalizing if every term is weakly normalizing. $\mathcal{R}$ is terminating (strongly normalizing) if every term is strongly normalizing.

$\mathbb{N}_k$ is an initial segment $\{1, 2, \ldots k\}$ of the set of positive numbers, where $\mathbb{N}_0 = \emptyset$. $\mathcal{P}(\mathbb{N})$ is the powerset of natural numbers.

3 Context-sensitive rewriting

A mapping $\mu : \Sigma \to \mathcal{P}(\mathbb{N})$ is a *replacement map* (or Σ-map) for the signature Σ iff for all $f \in \Sigma$. $\mu(f) \subseteq \mathbb{N}_{ar(f)}$. Thus, $\mu(f)$ are the *argument* positions which can be reduced for each $f \in \Sigma$ [8]. The ordering $\subseteq$ on $\mathcal{P}(\mathbb{N})$ extends pointwise to an ordering $\sqsubseteq$ on M_Σ, the set of all Σ-maps: $\mu \sqsubseteq \mu'$ if for all $f \in \Sigma$, $\mu(f) \subseteq \mu'(f)$. Then, $(M_\Sigma, \sqsubseteq, \mu_\bot, \mu_\top, \sqcup, \sqcap)$ is a complete lattice where $\mu_\bot(f) = \emptyset$, $\mu_\top(f) = \mathbb{N}_{ar(f)}$, $(\mu \sqcup \mu')(f) = \mu(f) \cup \mu'(f)$ and $(\mu \sqcap \mu')(f) = \mu(f) \cap \mu'(f)$ for all $f \in \Sigma$. Thus, $\mu \sqsubseteq \mu'$ means that μ considers less positions than μ' for reduction.

The set of μ-*replacing* occurrences $O^\mu(t)$ of t is defined by: $O^\mu(x) = \{\epsilon\}$ if $x \in V$, $O^\mu(f(t_1, \ldots, t_k)) = \{\epsilon\} \cup \bigcup_{i \in \mu(f)} i.O^\mu(t_i)$. Denote as $\widetilde{O^\mu}(t) = O(t) \backslash O^\mu(t)$, the set of *non-replacing* occurrences. Denote as $Var^\mu(t)$ the set of *replacing variables* of a term t, i.e. $x \in Var^\mu(t) \Leftrightarrow \exists u \in O^\mu(t).\ t|_u = x$. Also, $\widetilde{Var^\mu}(t) = Var(t) \backslash Var^\mu(t)$.

In *csr*, we rewrite *replacing* occurrences: a term t μ-rewrites to s, written $t \overset{u}{\hookrightarrow}_{\mathcal{R}(\mu)} s$ (or just $t \hookrightarrow s$), if $t \overset{u}{\to}_{\mathcal{R}} s$ and $u \in O^\mu(t)$. The set of *replacing redexes* is $O^\mu_{\mathcal{R}}(t) = O_{\mathcal{R}}(t) \cap O^\mu(t)$. The usual computational properties for terms t and TRSs $\mathcal{R}$ in rewriting (normal form, weak normalization, termination, etc) defined in terms of $O_{\mathcal{R}}(t)$ and $\to_{\mathcal{R}}$, generalize to *csr* (μ-normal form, weak μ-normalization, μ-termination, etc) using $O^\mu_{\mathcal{R}}(t)$ and $\hookrightarrow_{\mathcal{R}(\mu)}$.

Note that for the (top) Σ-map $\mu_\top$, $O^{\mu_\top}(t) = O(t)$. This is to say that, given a TRS $\mathcal{R}$, *csr* for $\mu_\top$ and unrestricted rewriting coincide, i.e. $\hookrightarrow_{\mathcal{R}(\mu_\top)} = \to_{\mathcal{R}}$.

A term t is μ-*compatible* ($\mathsf{comp}_\mu(t)$), if the non-replacing occurrences of t are variables: $\mathsf{comp}_\mu(t)$ iff $\widetilde{O^\mu}(t) \subseteq O_V(t)$. Equivalently, if any non-variable occurrence of t is a replacing occurrence: $\mathsf{comp}_\mu(t)$ iff $O_\Sigma(t) \subseteq O^\mu(t)$. comp is extended to sets of terms: $\mathsf{comp}_\mu(T) \Leftrightarrow \forall t \in T. \mathsf{comp}_\mu(t)$.

Given a term t, we can associate a *minimum* $\Sigma(t)$-map μ_t ensuring that t is compatible: if $t \in V$, then $\mu_t = \mu_\bot$. If $t = f(t_1, \ldots, t_k)$, we define the auxiliary $\Sigma(t)$-map μ_t^ϵ to be $\mu_t^\epsilon(f) = \{i \mid t_i \notin V,\ 1 \le i \le ar(f)\}$ and $\mu_t^\epsilon(g) = \emptyset$ for all $g \neq f$. Then $\mu_t = \mu_t^\epsilon \sqcup \mu_{t_1} \sqcup \cdots \sqcup \mu_{t_k}$. For instance, given the term $t = \mathtt{first}(\mathtt{0}, \mathtt{x})$, we have $\mu_t(\mathtt{first}) = \{1\}$. Since $\mu_{\mathtt{s}(\mathtt{x})}(\mathtt{s}) = \emptyset$ and $\mu_{\mathtt{y}::\mathtt{z}}(::) = \emptyset$, for $t' = \mathtt{first}(\mathtt{s}(\mathtt{x}), \mathtt{y} :: \mathtt{z})$, we get $\mu_{t'}(\mathtt{s}) = \mu_{t'}(::) = \emptyset$ and $\mu_{t'}(\mathtt{first}) = \{1, 2\}$. Given a set of terms $T \subseteq \mathcal{T}(\Sigma, V)$, $\mu_T = \sqcup_{t \in T} \mu_t$, is the minimum replacement map which makes T compatible.

Given a TRS $\mathcal{R} = (\Sigma, R)$ the *canonical* replacement map $\mu^{com}_{\mathcal{R}}$ of $\mathcal{R}$, is $\mu^{com}_{\mathcal{R}} = \mu_{L(\mathcal{R})}$. Note that $\mu^{com}_{\mathcal{R}}$ makes compatible each *lhs* of $\mathcal{R}$.

Example 3.1 *Let us consider the TRS $\mathcal{R}$ which is mainly borrowed from [12].*

$\mathtt{primes} \to \mathtt{sieve}(\mathtt{from}(\mathtt{s}(\mathtt{s}(\mathtt{0}))))$	$\mathtt{if}(\mathtt{true}, \mathtt{x}, \mathtt{y}) \to \mathtt{x}$
$\mathtt{from}(\mathtt{x}) \to \mathtt{x} :: \mathtt{from}(\mathtt{s}(\mathtt{x}))$	$\mathtt{if}(\mathtt{false}, \mathtt{x}, \mathtt{y}) \to \mathtt{y}$
$\mathtt{sel}(\mathtt{0}, \mathtt{x} :: \mathtt{y}) \to \mathtt{x}$	$\mathtt{0} + \mathtt{x} \to \mathtt{x}$
$\mathtt{sel}(\mathtt{s}(\mathtt{x}), \mathtt{y} :: \mathtt{z}) \to \mathtt{sel}(\mathtt{x}, \mathtt{z})$	$\mathtt{s}(\mathtt{x}) + \mathtt{y} \to \mathtt{s}(\mathtt{x} + \mathtt{y})$
$\mathtt{sieve}(\mathtt{x} :: \mathtt{y}) \to \mathtt{x} :: \mathtt{sieve}(\mathtt{filter}(\mathtt{x}, \mathtt{y}))$	
$\mathtt{filter}(\mathtt{x}, \mathtt{y} :: \mathtt{z}) \to \mathtt{if}(\mathtt{x} \mid \mathtt{y}, \mathtt{filter}(\mathtt{x}, \mathtt{z}), \mathtt{y} :: \mathtt{filter}(\mathtt{x}, \mathtt{z}))$	

Assume that $\mid$ is a built-in predicate and $x|y$ means x divides *y. We have*

$\mu_{\mathcal{R}}^{com}(f) = \emptyset$ *for all* $f \in \{\mathtt{0}, \mathtt{s}, [], ::, \mathtt{primes}, \mathtt{from}\}$*,* $\mu_{\mathcal{R}}^{com}(+) = \mu_{\mathcal{R}}^{com}(\mathtt{sieve}) = \mu_{\mathcal{R}}^{com}(\mathtt{if}) = \{1\}$*,* $\mu_{\mathcal{R}}^{com}(\mathtt{filter}) = \{2\}$ *and* $\mu_{\mathcal{R}}^{com}(\mathtt{sel}) = \{1, 2\}$*.*

The following theorem illustrates the relevance of the replacement map $\mu_{\mathcal{R}}^{com}$. It expresses that *csr* is complete when computing root-stable terms.

Theorem 3.2 ([8]) *Let* $\mathcal{R} = (\Sigma, R)$ *be an almost orthogonal TRS and* μ *be a* Σ*-map such that* $\mu_{\mathcal{R}}^{com} \sqsubseteq \mu$*. Let* t *be a term and* $s = f(\tilde{s})$ *be a root-stable term. If* $t \to^* s$*, there exists a root-stable term* $s' = f(\widetilde{s'})$ *such that* $t \hookrightarrow^* s'$*, and* $s' \to^* s$*.*

Example 3.3 *Let* $\mathcal{R}$ *be as in Example 3.1. Then,* $\mathtt{primes}$ *computes the (infinite) list of prime numbers. We obtain the second one by evaluating* $\mathtt{sel}(1, \mathtt{primes})$ *(we use* n *as an abbreviate of* $\mathtt{s}^n(0)$*). We use* $\mu_{\mathcal{R}}^{com}$*.*

$$
\begin{aligned}
&\mathtt{sel}(1, \underline{\mathtt{primes}}) \hookrightarrow \mathtt{sel}(1, \mathtt{sieve}(\underline{\mathtt{from}(2)}))\\
&\quad \hookrightarrow \mathtt{sel}(1, \underline{\mathtt{sieve}(2 :: \mathtt{from}(3))})\\
&\quad \hookrightarrow \underline{\mathtt{sel}(1, 2 :: \mathtt{sieve}(\mathtt{filter}(2, \mathtt{from}(3))))}\\
&\quad \hookrightarrow \mathtt{sel}(0, \mathtt{sieve}(\mathtt{filter}(2, \underline{\mathtt{from}(3)})))\\
&\quad \hookrightarrow \mathtt{sel}(0, \mathtt{sieve}(\underline{\mathtt{filter}(2, 3 :: \mathtt{from}(4))}))\\
&\quad \hookrightarrow \mathtt{sel}(0, \mathtt{sieve}(\mathtt{if}(\underline{2|3}, \mathtt{filter}(\mathtt{x}, \mathtt{z}), 3 :: \mathtt{filter}(2, \mathtt{from}(4)))))\\
&\quad \hookrightarrow \mathtt{sel}(0, \mathtt{sieve}(\underline{\mathtt{if}(\mathtt{false}, \mathtt{filter}(\mathtt{x}, \mathtt{z}), 3 :: \mathtt{filter}(2, \mathtt{from}(4)))}))\\
&\quad \hookrightarrow \mathtt{sel}(0, \underline{\mathtt{sieve}(3 :: \mathtt{filter}(2, \mathtt{from}(4)))})\\
&\quad \hookrightarrow \underline{\mathtt{sel}(0, 3 :: \mathtt{filter}(2, \mathtt{from}(4)))}\\
&\quad \hookrightarrow 3
\end{aligned}
$$

Given $\mathcal{R} = (\Sigma, R)$, we can consider the signature Σ as the disjoint union $\Sigma = \mathcal{C} \uplus \mathcal{F}$ of symbols $c \in \mathcal{C}$, called *constructors*, having no associated rule, and symbols $f \in \mathcal{F}$, called *defined functions*, which are defined by some rule $f(\tilde{l}) \to r \in R$. Any TRS can be presented in this way by taking $\mathcal{F} = \{root(l) \mid l \to r \in R\}$ and $\mathcal{C} = \Sigma \backslash \mathcal{F}$. Constructor terms (*values*) are denoted $\delta \in \mathcal{T}(\mathcal{C}, V)$. Note that no constructor discipline is assumed.

Regarding normalization of terms, if we restrict ourselves to values, *csr* is powerful enough to compute them. Given a set $\mathcal{B} \subseteq \mathcal{C}$, the replacement map $\mu_{\mathcal{R}}^{\mathcal{B}}$ is $\mu_{\mathcal{R}}^{\mathcal{B}} = \mu_{\mathcal{R}}^{com} \sqcup \mu_{\mathcal{B}}$, where $\mu_{\mathcal{B}}(c) = \mathbb{N}_{ar(c)}$ for all $c \in \mathcal{B}$, and $\mu_{\mathcal{B}}(f) = \emptyset$ if $f \notin \mathcal{B}$.

Theorem 3.4 ([8]) *Let* $\mathcal{R} = (\Sigma, R) = (\mathcal{C} \uplus \mathcal{F}, R)$ *be a left-linear TRS,* $\mathcal{B} \subseteq \mathcal{C}$ *and* μ *be a* Σ*-map such that* $\mu_{\mathcal{R}}^{\mathcal{B}} \sqsubseteq \mu$*. Let* $t \in \mathcal{T}(\Sigma, V)$*, and* $\delta \in \mathcal{T}(\mathcal{B}, V)$*. Then,* $t \to^* \delta$ *iff* $t \hookrightarrow^* \delta$*.*

For instance, since the value of $\mathtt{sel}(1, \mathtt{primes})$ is a natural number (this can be deduced from the type which is usually associated with the symbol functions, see [8]), we define $\mathcal{B} = \{\mathtt{0}, \mathtt{s}\}$. Then, we have[2]: $\mu_{\mathcal{R}}^{\mathcal{B}}(::) = \mu_{\mathcal{R}}^{\mathcal{B}}(\mathtt{from}) = \emptyset$, $\mu_{\mathcal{R}}^{\mathcal{B}}(\mathtt{s}) = \mu_{\mathcal{R}}^{\mathcal{B}}(+) = \mu_{\mathcal{R}}^{\mathcal{B}}(\mathtt{sieve}) = \mu_{\mathcal{R}}^{\mathcal{B}}(\mathtt{if}) = \{1\}$, $\mu_{\mathcal{R}}^{\mathcal{B}}(\mathtt{filter}) = \{2\}$, $\mu_{\mathcal{R}}^{\mathcal{B}}(\mathtt{sel}) = \{1, 2\}$. In general, we are not able to compute arbitrary normal forms.

[2] Note that, in Example 3.3, we were able to compute the value by simply using $\mu_{\mathcal{R}}^{com}$. However, this is due to the particular definition of this TRS. For instance, by changing the $\mathtt{primes}$-rule to $\mathtt{primes} \to \mathtt{from}(\mathtt{s}(\mathtt{0}) + \mathtt{s}(\mathtt{0}))$ in the TRS of Example 3.1 we really do need to use $\mu_{\mathcal{R}}^{\mathcal{B}}$ to compute the value.

Example 3.5 *Let us consider the TRS:*

$\mathtt{h}(\mathtt{0}) \to \mathtt{0}$ $\qquad\qquad$ $\mathtt{g}(\mathtt{0},\mathtt{x}) \to \mathtt{0}$

$\mathtt{h}(\mathtt{s}(\mathtt{0})) \to \mathtt{0}$

Let $\mu(\mathtt{g}) = \mu(\mathtt{h}) = \mu(\mathtt{s}) = \{1\}$. $\mathtt{g}(\mathtt{s}(\mathtt{0}),\mathtt{h}(\mathtt{0}))$ *is a* μ*-normal form, but it is not a normal form, since* $\mathtt{g}(\mathtt{s}(\mathtt{0}),\underline{\mathtt{h}(\mathtt{0})}) \to \mathtt{g}(\mathtt{s}(\mathtt{0}),\mathtt{0})$*. Note that* $\mathtt{g}(\mathtt{s}(\mathtt{0}),\mathtt{0}) \notin \mathcal{T}(\mathcal{C},V)$*.*

Of course, when we speak about *completeness*, we mean that *csr* is able to compute interesting information (root-stable forms, values) using a non-trivial replacement map μ, i.e., $\mu \sqsubset \mu_\top$. Otherwise, the discussion is meaningless, because if $\mu = \mu_\top$, *csr* and unrestricted rewriting coincide.

3.1 Strongly replacing-independent and μ-left-normal TRSs

In this section, we introduce the class of *strongly replacing-independent* TRSs, for which we will prove that any $\mu_{\mathcal{R}}^{com}$-reduction is needed, and the μ*-left-normal* TRSs for which we prove that the (restricted) leftmost-outermost strategy is optimal and root-normalizing. We start with some preparatory definitions and properties: A term t is strongly μ-compatible ($\mathsf{scomp}_\mu(t)$) if every (and only) variable occurrences are non-replacing: $\mathsf{scomp}_\mu(t)$ iff $\widetilde{O^\mu}(t) = O_V(t)$. Alternatively $\mathsf{scomp}_\mu(t)$ iff $O_\Sigma(t) = O^\mu(t)$. Clearly $\mathsf{scomp}_\mu(t) \Rightarrow \mathsf{comp}_\mu(t)$. We extend the definition to sets of terms $T \subseteq \mathcal{T}(\Sigma,V)$: $\mathsf{scomp}_\mu(T) \Leftrightarrow \forall t \in T.\mathsf{scomp}_\mu(t)$. If $\mathsf{scomp}_\mu(T)$, we say that T is a strongly compatible set.

There are terms having *no* replacement map μ satisfying $\mathsf{scomp}_\mu(t)$. For instance, there is no replacement map which makes $\mathtt{f}(\mathtt{f}(\mathtt{x},\mathtt{a}),\mathtt{y})$ strongly compatible. Given a term t, only μ_t which can make t strongly compatible.

Proposition 3.6 *Let* $t \in \mathcal{T}(\Sigma,V)$ *and* μ *be a* $\Sigma(t)$*-map. Then,* $\mathsf{scomp}_\mu(t) \Rightarrow \mu = \mu_t$*.*

This proposition justifies that we do not make μ explicit in speaking about strongly compatible terms.

Also note a crucial difference between the extension of the properties comp and scomp from terms to sets of terms: A set T of terms which verifies $\mathsf{comp}_{\mu_t}(t)$ for each $t \in T$, immediately satisfies $\mathsf{comp}_{\mu_T}(T)$, where $\mu_T = \sqcup_{t\in T}\mu_t$. However, this is not the case with scomp.

Example 3.7 *Let* $t = \mathtt{first}(\mathtt{0},\mathtt{x})$ *and* $t' = \mathtt{first}(\mathtt{s}(\mathtt{x}),\mathtt{y} :: \mathtt{z})$*. Each term* t *and* t' *is strongly compatible w.r.t.* μ_t *and* $\mu_{t'}$*, respectively. However, the set* $\{t,t'\}$ *is not strongly compatible w.r.t.* $\mu_t \sqcup \mu_{t'}$*.*

The unique (if any) $\Sigma(T)$-map μ which makes a set T strongly compatible is μ_T. However, we note that replacement restrictions for symbols in $\Sigma\backslash\Sigma(T)$ can be freely established without damaging the strong compatibility of T. We assume this fact in the remainder of our exposition with no explicit mention.

Definition 3.8 (strongly replacing-independent TRS) *A TRS* $\mathcal{R}$ *is* strongly replacing-independent*, if* $L(\mathcal{R})$ *is a strongly compatible set of terms.*

Note that, if $\mathcal{R}$ is strongly replacing-independent, the canonical replacement map $\mu_{\mathcal{R}}^{com}$ is the only one which satisfies the property. Thus, $\mu_{\mathcal{R}}^{com}$ is the *intended* replacement map when dealing with strongly replacing-independent TRSs. An equivalent characterization is: $\mathcal{R}$ is strongly replacing-independent iff, for all $l \in L(\mathcal{R})$, we have $Var^{\mu_{\mathcal{R}}^{com}}(l) = \emptyset$. The TRS in Example 3.1 is strongly replacing-independent. The interesting property of these TRSs is that every replacing redex in non-root-stable terms is root-needed, as we show below.

Left-normal TRSs are well-known from [13] because the leftmost-outermost reduction strategy is normalizing for this class of TRSs. A TRS is left-normal if, in each *lhs* of the TRS, the function symbols *precede* the variable symbols when considering the ordering $\leq_L$. We generalize this notion.

Definition 3.9 (μ-left-normal TRS) *Given a replacement map μ, a TRS $\mathcal{R}$ is* μ-left-normal *if, for each $l \in L(\mathcal{R})$, there exists $u_l \in O(l)$ such that, for all $u \in O^{\mu}(l)$, $u \leq_L u_l \Rightarrow u \in O_{\Sigma}(l)$ and $u >_L u_l \Rightarrow u \in O_V(l)$.*

The `if`-rules or the `sieve`-rule of $\mathcal{R}$ in Example 3.1 are left-normal. Nevertheless, $\mathcal{R}$ is not left-normal due to the `filter`-rule and `sel`-rules. Any strongly replacing-independent TRS is $\mu_{\mathcal{R}}^{com}$-left-normal (we just take $u_l = max_{\leq_L}(O(l))$ for each rule $l \to r$). However, we have specific examples of this class of TRSs.

Example 3.10 *Let us consider the TRS $\mathcal{R}$:*

$\mathtt{first}(\mathtt{0}, \mathtt{x}) \to []$

$\mathtt{first}(\mathtt{s}(\mathtt{x}), \mathtt{y} :: \mathtt{z}) \to \mathtt{y} :: \mathtt{first}(\mathtt{x}, \mathtt{z})$

The lhs's of this TRS do not constitute a strongly compatible set of terms (see Example 3.7). We have $\mu_{\mathcal{R}}^{com}(\mathtt{s}) = \mu_{\mathcal{R}}^{com}(::) = \emptyset$ and $\mu_{\mathcal{R}}^{com}(\mathtt{first}) = \{1, 2\}$. This TRS is not strongly replacing-independent due to the first rule. It is not left-normal due to the second rule. However, it is $\mu_{\mathcal{R}}^{com}$-left-normal.

If $\mathcal{R}$ is μ-left-normal, and $\mu' \sqsubseteq \mu$, then $\mathcal{R}$ is μ'-left-normal. This is immediate, since $O^{\mu'}(l) \subseteq O^{\mu}(l)$ for all $l \in L(\mathcal{R})$. If we take $\mu = \mu_{\top}$, we obtain the usual definition of left-normal TRS. Therefore, every left-normal TRS is μ-left-normal w.r.t. any replacement map μ (since $\mu \sqsubseteq \mu_{\top}$). If $\mu = \mu_{\bot}$, then any TRS is μ-left-normal, because, for all *lhs* l, $O^{\mu_{\bot}}(l) = \{\epsilon\}$, and $\epsilon \in O_{\Sigma}(l)$ (because $l \notin V$). Thus, we take $u_l = \epsilon$. As we show below, the good properties arise when $\mu_{\mathcal{R}}^{com} \sqsubseteq \mu$.

4 Context-sensitive rewriting and needed rewriting

A *needed redex* in a term t is a redex which must be reduced (either itself or some descendant) in any normalizing derivation starting from t [4, 7].

Since for general, orthogonal TRSs, needed redexes are not computable, the notion of *necessary set of redexes* by Sekar and Ramakrishnan [14] gives a meaningful notion of neededness for almost orthogonal TRSs. A necessary set of redexes is a set of redexes such that, at least one of the redexes in this set, or some descendant, must always be reduced in each normalizing derivation. For orthogonal, constructor-based TRSs, they give an algorithm to compute necessary sets of redexes, thus defining a computable (parallel) normalization procedure.

Since *csr* is complete w.r.t. root-normalization of terms (Theorem 3.2) but not w.r.t. normalization of terms (Example 3.5), *needed* rewriting is not completely adequate for characterizing the *cs*-computations. The notion of *root-needed* computation has also been analyzed in the literature [5, 12]. We follow the formalization of Middeldorp [12] to develop our results. A redex in a term t is root-needed, if it is contracted, itself or a descendant, in every rewrite sequence from t to a root-stable form. Since root-stable terms are an intermediate step to reach a normal form, every root-needed redex is also needed. Any non-root-stable term has a root-needed redex. Root-needed redexes in maximal non-root-stable subterms of a term are needed. The notion of necessary set of redexes extends to root-necessary set of redexes in the obvious way. A strategy is hyper root-normalizing if, given a term t having a root-stable form, there are no rewrite sequences starting from t which contain infinitely many root-needed steps. Repeated contraction of root-necessary sets of redexes is said *root-necessary reduction.*

Theorem 4.1 ([12]) *Root-necessary reduction is hyper root-normalizing for almost orthogonal TRSs.*

Since a root-needed redex is a particular case of a root-necessary set of redexes, we have the following immediate consequence.

Theorem 4.2 ([12]) *Root-needed reduction is hyper root-normalizing for orthogonal TRSs.*

4.1 Root-stable neededness of context-sensitive computations

In this section, we compare the context-sensitive computations with root-needed reductions. In the sequel, we usually identify redexes in a term with its occurrences. Ideally, if $\mathcal{I}^r_{\mathcal{R}}(t)$ is the set of occurrences of root-needed redexes of a term t, then we would like to have $\mathcal{I}^r_{\mathcal{R}}(t) = O^{\mu}_{\mathcal{R}}(t)$ for some replacement map μ. Of course, this is very difficult to obtain, as the replacement restrictions imposed by means of a replacement map μ are very simple. However, we show that under some conditions, we can extract a subset of replacing redexes $I \subseteq O^{\mu}_{\mathcal{R}}(t)$ which is contained in $\mathcal{I}^r_{\mathcal{R}}(t)$ or, at least, which is a root-necessary set of redexes. Hence, it is possible to use the (simple) replacement restrictions to obtain more refined computations (i.e., being more *root-needed*).

Our first result corresponds to Theorem 4.3 in [12]: *'In an orthogonal TRS, any non-root-stable term has a root-needed redex'.*

Theorem 4.3 *Let $\mathcal{R}$ be an orthogonal TRS and μ be such that $\mu^{com}_{\mathcal{R}} \sqsubseteq \mu$. Every non-root-stable term has a μ-replacing root-needed redex.*

Theorem 4.3 means that, if we consider a replacement map μ such that $\mu^{com}_{\mathcal{R}} \sqsubseteq \mu$, then by restricting the reductions to μ-replacing occurrences we do *not* lose the possibility of reducing root-needed redexes.

Theorem 4.4 *Let $\mathcal{R}$ be an orthogonal TRS, μ be such that $\mu^{com}_{\mathcal{R}} \sqsubseteq \mu$ and t be a non-root-stable term. Then, $minimal(O^{\mu}_{\mathcal{R}}(t))$ is a root-necessary set of redexes.*

This result improves the use of replacement restrictions by Maranget [11] to achieve optimal derivations by using parallel rewriting in a graph reduction framework. Due to the need to work in a graph reduction framework, Maranget restricts himself to a smaller class of orthogonal TRSs, for which $Var^{\mu}(r) \subseteq Var^{\mu}(l)$ and $\widetilde{Var}^{\mu}(r) \subseteq \widetilde{Var}^{\mu}(l)$ for all rules $l \to r$ in the TRS. For instance, the $\mathtt{if}$-rules in Example 3.1 only satisfy these restrictions if $\mu(\mathtt{if}) = \{1,2,3\}$, i.e., no replacement restrictions are imposed. Thus, our results on neededness of restricted reductions are more general.

In strongly replacing-independent TRSs, given a term t, the ordered set $(O_{\mathcal{R}}^{\mu_{\mathcal{R}}^{com}}(t), \leq)$ is *flat*. This is due to the following fact.

Proposition 4.5 *Let $\mathcal{R}$ be an almost non-ambiguous, strongly replacing independent TRS and t be a term. Then, $minimal(O_{\mathcal{R}}^{\mu_{\mathcal{R}}^{com}}(t)) = maximal(O_{\mathcal{R}}^{\mu_{\mathcal{R}}^{com}}(t)) = O_{\mathcal{R}}^{\mu_{\mathcal{R}}^{com}}(t)$.*

Let us consider the derivation in Example 3.3. It is easy to verify that, in each step, the reduced redex is the only one which can be considered by *csr* to perform reductions[3]. When considering orthogonal, strongly replacing-independent TRSs, we get a surprising result: any replacing redex is a root-needed redex.

Theorem 4.6 *Let $\mathcal{R}$ be an orthogonal, strongly replacing-independent TRS. Let t be a non-root-stable term. Then, $O_{\mathcal{R}}^{\mu_{\mathcal{R}}^{com}}(t)$ is a set of root-needed redexes.*

We note that this means that, for every term t, $O_{\mathcal{R}}^{\mu_{\mathcal{R}}^{com}}(t)$ is a set of *needed* redexes. For instance, the derivation in Example 3.3 is (root-)needed reduction. However, there are root-needed redexes which are not replacing.

Example 4.7 *Let us consider the strongly replacing-independent TRS:*

$\mathsf{g}(\mathsf{a},\mathtt{x}) \to \mathtt{x}$ $\qquad$ $\mathsf{b} \to \mathsf{a}$

and the whole set of root-normalizing derivations for $\mathsf{g}(\mathsf{a},\mathsf{b})$*:*

1) $\mathsf{g}(\mathsf{a},\underline{\mathsf{b}}) \to \underline{\mathsf{g}(\mathsf{a},\mathsf{a})} \to \mathsf{a}$

2) $\underline{\mathsf{g}(\mathsf{a},\mathsf{b})} \to \underline{\mathsf{b}} \to \mathsf{a}$

As we can show, redex b *at the occurrence* 2 *of* $\mathsf{g}(\mathsf{a},\mathsf{b})$ *is root-needed. However,* 2 *is not a replacing occurrence.*

Thus, even in strongly replacing-independent TRS, we cannot obtain $\mathcal{I}_{\mathcal{R}}^{r}(t) = O_{\mathcal{R}}^{\mu}(t)$. Theorem 4.6 shows that it is possible to perform root-needed reduction in parallel. This makes Kennaway's claim in [5] unaccurate: *in an orthogonal TRS, no term can have more than one strongly root-needed redex.* Kennaway argues that this justifies the existence of a 'conflict' between call by need and parallelism. Our notion of strongly replacing-independent TRSs reveals that this conflict depends on the election of the computable approximation to root-needed redexes (strongly root-needed redexes[4] in [5]). For instance, by considering $\mathcal{R}$

[3] This is an extreme case of Proposition 4.5: $O_{\mathcal{R}}^{\mu_{\mathcal{R}}^{com}}(t)$ is a singleton

[4] In a recent work, Durand and Middeldorp [3] have shown some inconsistencies in Kennaway's approximation to root-neededness.

as in Example 3.1, the term $t = \mathtt{sel}(0 + 0, \mathtt{from}(0))$ is not root-stable, and we have $O_{\mathcal{R}}^{\mu_{\mathcal{R}}^{com}}(t) = \{1, 2\}$. By Theorem 4.6, 1 and 2 are occurrences of root-needed redexes. Thus, they can both be reduced in parallel.

We can still characterize root-needed redexes by means of replacement restrictions in the class of μ-left-normal TRSs.

Theorem 4.8 *Let $\mathcal{R}$ be an orthogonal TRS. Let μ be such that $\mu_{\mathcal{R}}^{com} \sqsubseteq \mu$ and $\mathcal{R}$ is μ-left-normal. Let t be a non-root-stable term. Then $min_{\leq_L}(O_{\mathcal{R}}^{\mu}(t))$ is a root-needed redex.*

5 Strategies in context-sensitive rewriting

A sequential rewriting strategy is a function $\mathbb{F}_{\mathcal{R}} : \mathcal{T}(\Sigma, V) \to \mathcal{T}(\Sigma, V)$ such that $\mathbb{F}_{\mathcal{R}}(t) = t$ if t is a normal form (i.e., $O_{\mathcal{R}}(t) = \emptyset$), and $t \to_{\mathcal{R}} \mathbb{F}_{\mathcal{R}}(t)$ otherwise (for parallel strategies $t \to_{\mathcal{R}}^{+} \mathbb{F}_{\mathcal{R}}(t)$) [6].

We drop the suffix $\mathcal{R}$ when no confusion arises. Also, we write $t \to_{\mathbb{F}} s$ instead of $t \to_{\mathcal{R}} \mathbb{F}_{\mathcal{R}}(t)$ if $s = \mathbb{F}_{\mathcal{R}}(t)$. The result of applying a sequential strategy $\mathbb{F}$ for a given TRS $\mathcal{R} = (\Sigma, R)$ can be split into two components: $\mathbb{P} : \mathcal{T}(\Sigma, V) \to \mathbb{N}^* \cup \{\Box\}$ which selects the redex occurrence which should be reduced, and $\mathbb{R} : \mathcal{T}(\Sigma, V) \to R$ which selects the rule which must be applied (see [1]). Then,

$\mathbb{P}(t) = \Box$ iff t is a normal form.

If $u = \mathbb{P}(t) \neq \Box$, then $\mathbb{F}(t) = t[\sigma(r)]_u$ where $t|_u = \sigma(l)$, and $l \to r = \mathbb{R}(t)$.

Sometimes, $\mathbb{P}$ can be further broken down as follows: $\mathbb{P} = \widehat{\mathbb{P}} \circ O_{\mathcal{R}}$ where $\widehat{\mathbb{P}} : \mathcal{P}(\mathbb{N}^*) \to \mathbb{N}^* \cup \{\Box\}$ selects an occurrence from a non empty set of redex occurrences ($\widehat{\mathbb{P}}(\emptyset) = \Box$, and $\widehat{\mathbb{P}}(U) \subseteq U$ for any $U \in \mathcal{P}(\mathbb{N}^*)$), and $O_{\mathcal{R}} : \mathcal{T}(\Sigma, V) \to \mathcal{P}(\mathbb{N}^*)$ is the usual function which gives the redex occurrences of a term t. Similar considerations can be made for parallel strategies.

For instance, let us consider the following well-known rewriting strategies [6]. We show the decomposition $\mathbb{P} = \widehat{\mathbb{P}} \circ O_{\mathcal{R}}$ for each strategy.

1. The *leftmost-innermost* strategy ($\mathbb{F}_{li}$) reduces the leftmost of the innermost redexes: $\mathbb{P}_{li}(t) = min_{\leq_L}(maximal_{\leq}(O_{\mathcal{R}}(t)))$.
2. The *parallel-innermost* strategy ($\mathbb{F}_{pi}$) simultaneously reduces all innermost redexes: $\mathbb{P}_{pi}(t) = maximal_{\leq}(O_{\mathcal{R}}(t))$.
3. The *leftmost-outermost* strategy ($\mathbb{F}_{lo}$) reduces the leftmost of the outermost redexes: $\mathbb{P}_{lo}(t) = min_{\leq_L}(minimal_{\leq}(O_{\mathcal{R}}(t))) = min_{\leq_L}(O_{\mathcal{R}}(t))$.
4. The *parallel-outermost* strategy ($\mathbb{F}_{po}$) reduces all outermost redexes: $\mathbb{P}_{po}(t) = minimal_{\leq}(O_{\mathcal{R}}(t))$.
5. The *full substitution rule* ($\mathbb{F}_{fs}$): simultaneously reduces all redexes (only defined for orthogonal TRSs): $\mathbb{P}_{fs}(t) = O_{\mathcal{R}}(t)$.

Note that, in the previous strategies, the question of *what* rule applies to a selected redex is left open. In non-ambiguous TRSs this is not a problem, since only one rule per redex applies ($\mathbb{R}$ is given by $\mathbb{P}$).

A rewriting strategy $\mathbb{F}$ for a TRS $\mathcal{R}$ is *normalizing* if, for each t having a normal form, the sequence $(\mathbb{F}^n(t) \mid n \geq 0)$ contains a normal form. For orthogonal

TRSs, results on normalization by means of the previous strategies are due to O'Donnell [13] (see also [6]): $\mathbb{F}_{li}$ and $\mathbb{F}_{pi}$ are normalizing iff the TRS is terminating. $\mathbb{F}_{lo}$ is normalizing for left-normal TRSs. $\mathbb{F}_{po}$ and $\mathbb{F}_{fs}$ are normalizing.

5.1 Context-sensitive strategies and restricted strategies

The notion of context-sensitive strategy or μ-strategy $\mathbb{F}_{\mathcal{R}(\mu)}$ is analogous to the standard notion as defined above: $\mathbb{F}_{\mathcal{R}(\mu)}(t) = t$ if t is a μ-normal form (i.e., $O^{\mu}_{\mathcal{R}}(t) = \emptyset$) and $t \hookrightarrow_{\mathcal{R}(\mu)} \mathbb{F}_{\mathcal{R}(\mu)}(t)$, otherwise, ($t \hookrightarrow^{+}_{\mathcal{R}(\mu)} \mathbb{F}_{\mathcal{R}(\mu)}(t)$ for parallel μ-strategies). Given a strategy $\mathbb{F}$, and a replacement map μ, we define the associated μ-restricted μ-strategy.

Definition 5.1 (μ-restriction of a rewriting strategy) *Let $\mathcal{R} = (\Sigma, R)$ be an orthogonal TRS, and μ be a Σ-map. Let $\mathbb{F}$ be a rewriting strategy with components $\mathbb{P}$, $\mathbb{R}$. Let $\mathbb{P} = \widehat{\mathbb{P}} \circ O_{\mathcal{R}}$. The μ-restriction of $\mathbb{F}$ is the μ-strategy $\mathbb{F}^{\mu}$ having components $\mathbb{P}^{\mu} = \widehat{\mathbb{P}} \circ O^{\mu}_{\mathcal{R}}$ and $\mathbb{R}^{\mu}$, where $\mathbb{R}^{\mu}$ is (uniquely) determined by the redexes selected by $\mathbb{P}^{\mu}$.*

For the leftmost-outermost, parallel-innermost, ... rewriting strategies, we obtain the corresponding μ-restricted μ-strategies by considering the *μ-replacing* redexes in a term (and only these ones). We just take $O^{\mu}_{\mathcal{R}}(t)$ instead of $O_{\mathcal{R}}(t)$ in the previous expressions for $\mathbb{P}_{li}, \ldots, \mathbb{P}_{fs}$, thus giving $\mathbb{P}^{\mu}_{li}, \ldots, \mathbb{P}^{\mu}_{fs}$, for the corresponding μ-restricted μ-strategies $\mathbb{F}^{\mu}_{li}, \ldots, \mathbb{F}^{\mu}_{fs}$.

We analyze the properties of μ-restricted μ-strategies $\mathbb{F}^{\mu}_{li}, \ldots, \mathbb{F}^{\mu}_{fs}$ concerning root-normalization. In the next section, we discuss normalization. The notion of root-normalizing strategy is analogous to normalizing strategies. We note the very special situation which arises with strongly replacing-independent TRSs:

Theorem 5.2 *Let $\mathcal{R}$ be an orthogonal, strongly replacing-independent TRS. Then, every $\mu^{com}_{\mathcal{R}}$-strategy $\mathbb{F}_{\mathcal{R}(\mu^{com}_{\mathcal{R}})}$ is hyper root-normalizing.*

In particular, when considering orthogonal, strongly replacing-independent TRSs, Proposition 4.5 entails that innermost and outermost $\mu^{com}_{\mathcal{R}}$-strategies collapse into a single class of *cs*-strategies.

The restriction of parallel-outermost and full substitution strategies are also root-normalizing.

Theorem 5.3 *Let $\mathcal{R}$ be an orthogonal TRS, and μ be such that $\mu^{com}_{\mathcal{R}} \sqsubseteq \mu$. Then, $\mathbb{F}^{\mu}_{po}$ and $\mathbb{F}^{\mu}_{fs}$ are hyper root-normalizing.*

Note that, since in the μ-restricted μ-strategies we consider a lesser set of redexes (because $minimal(O^{\mu}_{\mathcal{R}}(t)) \subseteq minimal(O_{\mathcal{R}}(t))$ and $O^{\mu}_{\mathcal{R}}(t) \subseteq O_{\mathcal{R}}(t)$), we obtain some advantages w.r.t. the corresponding parallel strategies.

Theorem 5.4 *Let $\mathcal{R}$ be an orthogonal TRS, and μ be such that $\mu^{com}_{\mathcal{R}} \sqsubseteq \mu$ and $\mathcal{R}$ is μ-left-normal. Then $\mathbb{F}^{\mu}_{lo}$ is hyper root-normalizing.*

Example 5.5 *Let us consider the $\mu^{com}_{\mathcal{R}}$-left-normal TRS $\mathcal{R}$:*

$$\mathtt{g(f(a),f(x))} \to \mathtt{a} \qquad\qquad \mathtt{b} \to \mathtt{b}$$
$$\mathtt{h(x,a,f(y))} \to \mathtt{a}$$

where $\mu^{com}_{\mathcal{R}}(\mathtt{f}) = \{1\}$, $\mu^{com}_{\mathcal{R}}(\mathtt{h}) = \{2,3\}$, and $\mu^{com}_{\mathcal{R}}(\mathtt{g}) = \{1,2\}$. We have the leftmost-outermost infinite derivation:

$$\mathtt{g(f(h(\underline{b},h(x,a,f(y)),f(y))),f(b))} \to \mathtt{g(f(h(\underline{b},h(x,a,f(y)),f(y))),f(b))} \to \cdots$$

However, the corresponding $\mu^{com}_{\mathcal{R}}$-restricted leftmost-outermost derivation is:

$$\mathtt{g(f(h(b,\underline{h(x,a,f(y))},f(y))),f(b))} \hookrightarrow \mathtt{g(f(\underline{h(b,a,f(y))}),f(b))}$$
$$\hookrightarrow \underline{\mathtt{g(f(a),f(b))}} \hookrightarrow \mathtt{a}$$

which does root-normalize the term. However, we cannot use any *$\mu^{com}_{\mathcal{R}}$-restricted $\mu^{com}_{\mathcal{R}}$-strategy. For instance, the 'rightmost' $\mu^{com}_{\mathcal{R}}$-reduction:*

$$\mathtt{g(f(h(b,h(x,a,f(y)),f(y))),f(\underline{b}))} \hookrightarrow \mathtt{g(f(h(b,h(x,a,f(y)),f(y))),f(\underline{b}))} \hookrightarrow \cdots$$

is infinite.

When we consider left-normal TRSs, there is no difference between the leftmost-outermost strategy and the restricted version. This is because they select exactly the same occurrences for reductions.

Proposition 5.6 *Let $\mathcal{R}$ be a left-normal TRS, μ be such that $\mu^{com}_{\mathcal{R}} \sqsubseteq \mu$ and t be a non-root-stable term. Then, $\mathbb{P}_{lo}(t) = \mathbb{P}^{\mu}_{lo}(t)$.*

5.2 Normalization with context-sensitive strategies

In this section, we discuss the use of a μ-strategy $\mathbb{F}_{\mathcal{R}(\mu)}$ to *normalize* terms. We know that, in general, *csr* is not able to compute normal forms, but only μ-normal forms. However, we have the following possibilities:

1. $\mathbb{F}_{\mathcal{R}(\mu)}$ is root-normalizing. Then, we can use it to obtain a root-stable form s of a term t and recursively apply it to the immediate subterms of s [5, 12]. We call this, *normalization via root-normalization*. The problem is that, in general, it is undecidable whether a term is root-stable or not.
2. $\mathbb{F}_{\mathcal{R}(\mu)}$ is μ-normalizing. Then, we can use it to obtain a μ-normal form s of a term t. Then, we recursively apply the μ-strategy to maximal non-replacing subterms of s. We call this, *normalization via μ-normalization*. We only need to provide some additional machinery to perform the 'jumps' into non-replacing parts of μ-normal forms (since this is not pure *csr*).
3. $\mathbb{F}_{\mathcal{R}(\mu)}$ is μ-normalizing, and the μ-normal form of t is a normal form (for instance, a value). Then, we can use $\mathbb{F}_{\mathcal{R}(\mu)}$ with no further refinements (*direct normalization*).

The first possibility is fairly discussed in [12]. In Section 5.1, we have characterized some root-normalizing cs-strategies. However, as stated in [12], not every root-normalizing strategy is normalizing and vice-versa. We must restrict ourselves to *context-free* root-normalizing strategies.

Definition 5.7 ([12]) *A sequential reduction strategy $\mathbb{F}$ is said to be context-free if for all root-stable terms $t = f(t_1,\ldots,t_i,\ldots,t_k)$ and $i \in \{1,\ldots,k\}$, such that $t \to_{\mathbb{F}} f(t_1,\ldots,t'_i,\ldots,t_k)$, we have $t_i \to_{\mathbb{F}} t'_i$.*

Almost all strategies in the literature are context-free. The restricted *cs*-strategies in the previous section also are. We refer the reader to [12] for further details.

Concerning normalization via μ-normalization, a normalizing strategy is able to compute a normal form of a *weakly normalizing* term t. If we want to use a μ-normalizing μ-strategy $\mathbb{F}_{\mathcal{R}(\mu)}$ to obtain the normal form $\bar{t}$ of t, via a μ-normal form s, we must first ensure that the weakly normalizing term t is weakly μ-normalizing. Then, the μ-normalizing μ-strategy will compute s. In contrast to termination (see [9]), weak normalization is *not* preserved by *csr*.

Example 5.8 *Let us consider the (nonterminating) TRS*

$\mathtt{f(x,a)} \to \mathtt{a} \qquad \mathtt{g(a)} \to \mathtt{a} \qquad \mathtt{b} \to \mathtt{b}$

If we take $\mu(\mathtt{f}) = \{1\}$ *and* $\mu(\mathtt{g}) = \varnothing$*, then:*

1. *The term* $t = \mathtt{f(b,g(a))}$ *is weakly normalizing:* $\mathtt{f(b,\underline{g(a)})} \to \underline{\mathtt{f(b,a)}} \to \mathtt{a}$. *However,* t *is not weakly* μ*-normalizing, as there is only the following infinite reduction sequence:* $\mathtt{f(\underline{b},g(a))} \hookrightarrow \mathtt{f(\underline{b},g(a))} \hookrightarrow \cdots$.
2. *The term* $t = \mathtt{g(b)}$ *is weakly* μ*-normalizing (it is a* μ*-normal form), but* t *is not weakly normalizing:* $\mathtt{g(\underline{b})} \to \mathtt{g(\underline{b})} \to \cdots$.

However, when we work with the canonical replacement map (or greater), we have the following.

Theorem 5.9 *Let* $\mathcal{R}$ *be an almost orthogonal TRS and* μ *be such that* $\mu^{com}_{\mathcal{R}} \sqsubseteq \mu$*. If* t *is weakly normalizing, then* t *is weakly* μ*-normalizing.*

The next point is how to define the μ-normalizing μ-strategy $\mathbb{F}_{\mathcal{R}(\mu)}$. The connection between root-normalizing and μ-normalizing μ-strategies is also immediate. In fact, the proof is almost the same as Theorem 6.4 in [12] by considering the replacement restrictions.

Theorem 5.10 *Let* $\mathcal{R}$ *be a confluent TRS. A context-free root-normalizing* μ*-strategy is* μ*-normalizing.*

Since the restricted *cs*-strategies analyzed in the previous section are context-free, an easy consequence of Theorem 5.10 and Theorem 5.3 is that, for orthogonal TRSs, $\mathbb{F}^{\mu}_{po}$ and $\mathbb{F}^{\mu}_{fs}$ are μ-normalizing whenever $\mu^{com}_{\mathcal{R}} \sqsubseteq \mu$.

Theorem 4.6 says that any $\mu^{com}_{\mathcal{R}}$-reduction of a non-root-stable term in orthogonal, strongly replacing-independent TRSs only takes root-needed redexes. Thus, reductions to root-stable form performed by any $\mu^{com}_{\mathcal{R}}$-strategy in such TRSs are root-needed. Since root-needed reduction is trivially context-free (see [12]), Theorem 5.10 and Theorem 5.2 reveal that every root-normalizing $\mu^{com}_{\mathcal{R}}$-strategy is $\mu^{com}_{\mathcal{R}}$-normalizing for orthogonal, strongly replacing-independent TRSs. We have the following immediate consequence.

Theorem 5.11 *An orthogonal, strongly replacing-independent TRS* $\mathcal{R}$ *is* $\mu^{com}_{\mathcal{R}}$*-terminating iff it is weakly* $\mu^{com}_{\mathcal{R}}$*-normalizing.*

Finally, if $\mu_{\mathcal{R}}^{com} \sqsubseteq \mu$, by Theorem 5.10 and Theorem 5.4, we know that $\mathbb{F}_{lo}^{\mu}$ is μ-normalizing for orthogonal μ-left-normal TRSs.

Jumping into maximal non-replacing subterms of a μ-normal form is safe because replacing subterms of μ-normal forms are μ-normal forms and, provided that $\mu_{\mathcal{R}}^{com} \sqsubseteq \mu$, every μ-normal form is root-stable (see [8]).

For the last normalization procedure, whenever the normal form of a term is a value, we can use *csr* to compute it (Theorem 3.4). Now we are more concrete: we can use context-sensitive strategies to compute it.

Theorem 5.12 *Let $\mathcal{R} = (\Sigma, R) = (\mathcal{F} \uplus \mathcal{C}, R)$ be a confluent, left linear TRS. Let $\mathcal{B} \subseteq \mathcal{C}$ and $\delta \in \mathcal{T}(\mathcal{B}, V)$. Let μ be a Σ-map such that $\mu_{\mathcal{R}}^{\mathcal{B}} \sqsubseteq \mu$, and $\mathbb{F}_{\mathcal{R}(\mu)}$ be a μ-normalizing μ-strategy. Then $t \to^* \delta$ iff $t \hookrightarrow^*_{\mathbb{F}_{\mathcal{R}(\mu)}} \delta$.*

As opposed to Theorem 3.4, confluence is required in Theorem 5.12, because our strategies are deterministic. Otherwise, the value computed by $\mathbb{F}_{\mathcal{R}(\mu)}$ could be different from δ. Dealing with almost orthogonal TRSs, Theorem 5.9 shows that the computation of a normal form is consistent with the computation of a μ-normal form: if a term has a normal form, it also has a μ-normal form. Hence, if the input term has a normal form which is not a value, the μ-normalizing μ-strategy will stop (giving a μ-normal form). Thus, the computation is safe.

Even if we deal with μ-*terminating* TRSs [9, 16] the results in Section 4.1 ensure optimality of these *cs*-strategies in the sense that they reduce needed redexes. Since our optimality results concern $\mu_{\mathcal{R}}^{com}$ rather than $\mu_{\mathcal{R}}^{\mathcal{B}}$, we can lose optimality in context-sensitive reductions using $\mu_{\mathcal{R}}^{\mathcal{B}}$. To avoid this undesirable behavior, we have proposed a solution by using program transformations. They allow us to employ the results on neededness which are given in this paper to obtain complete evaluations which only consider needed redexes [10].

6 Conclusions and future work

We have given conditions to ensure the efficiency of computations using replacement maps greater than or equal to the canonical replacement map $\mu_{\mathcal{R}}^{com}$. We have also shown how to use the replacement restrictions to find out (root-)needed redexes. We have defined classes of TRSs, namely strongly replacing-independent and μ-left normal TRSs, for which this is very easy to achieve. We have introduced the notion of context-sensitive strategy. We have shown how to *restrict* a given rewriting strategy to obtain a *cs*-strategy. We have given conditions ensuring that the restricted *cs*-strategies obtained from the usual strategies preserve the good properties of the original ones. Restricted *cs*-strategies improve the behavior of the unrestricted ones by reducing the amount of wasteful computations or widening the class of TRSs for which the strategy is well-behaved. In this way, we obtain a suitable framework for the definition of normalizing strategies.

Completeness of *csr* concerns left-linear TRSs, rather than orthogonal ones (see [8]). A subject of future work could be to establish accurated approximations to root-needed and needed computations in a more general framework [3].

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