Termination of (Canonical) Context-Sensitive Rewriting

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Introduction

Consider a function call $f(t_1, \ldots, t_k)$

- A *lazy* strategy evaluates a given t_i, $1 \leq i \leq k$ if *necessary*.

  $(+)$ Improves termination. Unwasteful.

  $(-)$ Implementation is complex.

- An *eager* strategy first evaluates *each* t_i, $1 \leq i \leq k$.

  $(+)$ Easy to implement (and understand).

  $(-)$ Non-termination.

Introducing context-sensitive rewriting (*CSR*)

Given a function call $f(t_1, \ldots, t_k)$ we (only) evaluate the arguments indicated by $\mu(f) \subseteq \{1, \ldots, k\}$.

Example :

$$\texttt{if(true,x,y)} \to \texttt{x}$$

$$\texttt{if(false,x,y)} \to \texttt{y}$$

Given a call

$$\texttt{if(}cond\texttt{,}exp\texttt{,}exp'\texttt{)}$$

we avoid reductions on both exp and exp' if $\mu(\texttt{if}) = \{1\}$.

Using context-sensitive rewriting

The following TRS can be used to arbitrarily approximate $\pi^2/6$:

```
sqr(0)       → 0                       0 + x            → x
sqr(s(x))    → s(sqr(x)+dbl(x))        s(x) + y         → s(x+y)
dbl(0)       → 0                       first(0,x)       → []
dbl(s(x))    → s(s(dbl(x)))            first(s(x),y:z)  → y:first(x,z)
half(0)      → 0                       half(s(s(x)))    → s(half(x))
half(s(0))   → 0                       half(dbl(x))     → x

terms(n)     → recip(sqr(n)):terms(s(n))
```

No existing results describing normalizing strategies for left-linear (possibly overlapping) TRSs apply to $\mathcal{R}$ (!).

Using context-sensitive rewriting

$>>$ Can *CSR* be helpful in this case? $<<$

Yes ! $\Longrightarrow$

① Use the canonical replacement map

② Prove (canonical) termination of *CSR*

③ Take a term; compute the normal form w.r.t. *CSR*

④ Jump into its maximal non-replacing parts

⑤ Go to ③, if possible

Summary

① Basic description of (canonical) *CSR*

② Normalization via μ-normalization

③ Proving termination of canonical *CSR*

④ Termination or canonical μ-termination?

⑤ Conclusions and future work

Replacement maps and replacing positions

A mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ such that $\mu(f) \subseteq \{1, \ldots, k\}$ for every k-ary $f \in \mathcal{F}$, is called a replacement map or $\mathcal{F}$-map (Lucas [JFLP'98]).

The set of all $\mathcal{F}$-maps is $M_{\mathcal{F}}$ (or $M_{\mathcal{R}}$ if $\mathcal{F}$ comes from a TRS $\mathcal{R} = (\mathcal{F}, R)$)

The set of replacing positions is given by:

$\mathcal{P}os^{\mu}(x) = \{\epsilon\}$ if $x \in \mathcal{X}$

$\mathcal{P}os^{\mu}(f(\tilde{t})) = \{\epsilon\} \cup (\bigcup_{i \in \mu(f)} i.\mathcal{P}os^{\mu}(t_i))$

Maximal replacing context

Given a term t, *MRC*$^\mu(t)$ is the maximal prefix of t whose positions are μ-replacing in t.

Example Consider the following replacement map:

$$\mu(f) = \{1\}, \text{for } f \in \{\mathtt{s}, \mathtt{:}, \mathtt{dbl}, \mathtt{half}, \mathtt{recip}, \mathtt{sqr}, \mathtt{terms}, \mathtt{+}\}$$

$$\text{and } \mu(\mathtt{first}) = \{1, 2\}$$

For $t = \mathtt{recip(s(0)):first(s(0),terms(s(s(0))))}$, we have

$$\mathcal{P}os^\mu(t) = \{\epsilon,\ 1,\ 1.1,\ 1.1.1\} \quad \text{and} \quad \textit{MRC}^\mu(t) = \mathtt{recip(s(0)):}\Box$$

Context-sensitive rewriting

Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, and μ be a $\mathcal{F}$-map. In *CSR*, we only rewrite replacing redexes: t μ-rewrites to s, written

$$t \hookrightarrow_{\mathcal{R}(\mu)} s,$$

if $t \stackrel{p}{\to}_{\mathcal{R}} s$ and $p \in \mathcal{P}os^{\mu}(t)$.

Canonical replacement map

The *canonical replacement map* $\mu_{\mathcal{R}}^{can}$ for a TRS $\mathcal{R}$ is [JFLP'98]:

> *the most restrictive replacement map which ensures that the non-variable subterms of the left-hand sides of the rules of $\mathcal{R}$ are replacing*.

Let *CM*$_{\mathcal{R}}$ be the set of replacement maps which are less than or equally restrictive to $\mu_{\mathcal{R}}^{can}$.

Canonical replacement map

Consider the TRS $\mathcal{R}$:

$$\begin{array}{lcl} \texttt{first(0,x)} & \rightarrow & \texttt{[]} \\ \texttt{first(s(x),y:z)} & \rightarrow & \texttt{y:first(x,z)} \end{array} \qquad \texttt{from(x)} \rightarrow \texttt{x:from(s(x))}$$

we have

- $1 \in \mu_{\mathcal{R}}^{can}(\texttt{first})$ because, e.g., $\texttt{first(0,x)}|_1 = \texttt{0} \notin \mathcal{X}$; and
- $2 \in \mu_{\mathcal{R}}^{can}(\texttt{first})$ because $\texttt{first(s(x),y:z)}|_2 = \texttt{y:z} \notin \mathcal{X}$.

Therefore,

$$\mu_{\mathcal{R}}^{can}(\texttt{first}) = \{1,2\} \quad \text{and} \quad \mu_{\mathcal{R}}^{can}(\texttt{s}) = \mu_{\mathcal{R}}^{can}(\texttt{:}) = \mu_{\mathcal{R}}^{can}(\texttt{from}) = \varnothing$$

Computing head-normal forms

Theorem [JFLP'98] Let $\mathcal{R}$ be a left-linear TRS and $\mu \in$ *CM*$_{\mathcal{R}}$. Every μ-normal form is a head-normal form.

Theorem [IC'02] Let $\mathcal{R}$ be a left-linear TRS and $\mu \in$ *CM*$_{\mathcal{R}}$. If $t \to^! s$, then $t \hookrightarrow^!_\mu t' \to^! s$ for some term t'.

Corollary [IC'02] Let $\mathcal{R}$ be a left-linear TRS and $\mu \in$ *CM*$_{\mathcal{R}}$. Every normalizing term is μ-normalizing.

Computing normal forms

```
Procedure  norm_μ(T)
   T := μ-norm(T)
   for each t ∈ T
    let t = C[t_1, ..., t_n],  where C[ ] = MRC^μ(t)
    for i := 1, ..., n do S_i := norm_μ({t_i})
    T_t := C[S_1, ..., S_n]
   return ⋃_{t ∈ T} T_t
end procedure  norm_μ
```

Normalization via μ-normalization

We can obtain the first two terms of the infinite series converging to $\pi^2/6$ as *norm*$_\mu(\{\texttt{first(2,terms(1))}\})$ for μ as above:

$\texttt{first(2,}\underline{\texttt{terms(1)}}\texttt{)} \hookrightarrow \underline{\texttt{first(2,recip(sqr(1)):terms(2))}}$

$\hookrightarrow \texttt{recip(sqr(1)):first(1,terms(2))}$

$\hookrightarrow \texttt{recip(s(}\underline{\texttt{sqr(0)}}\texttt{+dbl(0))):first(1,terms(2))}$

$\hookrightarrow \texttt{recip(s(}\underline{\texttt{0+dbl(0)}}\texttt{)):first(1,terms(2))}$

$\hookrightarrow \texttt{recip(s(}\underline{\texttt{dbl(0)}}\texttt{)):first(1,terms(2))}$

$\hookrightarrow \texttt{recip(1):first(1,terms(2))}$

At this point, the computation stops yielding a μ-normal form

$$s = \texttt{recip(1):first(1,terms(2))}$$

Normalization via μ-normalization

... but, since $\mathit{MRC}^{\mu}(s) =$ `recip(1):`$\Box$, now we jump into subterm `first(1,terms(2))` of s:

```
recip(1):first(1,terms(2))
   → recip(1):first(1,recip(sqr(2)):terms(3))
   → recip(1):recip(sqr(2)):first(0,terms(3))
   → recip(1):recip(s(sqr(1)+dbl(1))):first(0,terms(3))
   → recip(1):recip(s(s(sqr(0)+dbl(0))+dbl(1))):first(0,terms(3))
   → recip(1):recip(s(s(sqr(0)+dbl(0)+dbl(1)))):first(0,terms(3))
   → recip(1):recip(s(s(0+dbl(0)+dbl(1)))):first(0,terms(3))
   → recip(1):recip(s(s(dbl(0)+dbl(1)))):first(0,terms(3))
```

$\rightarrow$ `recip(1):recip(s(s(0+dbl(1)))):first(0,terms(3))`

$\rightarrow$ `recip(1):recip(s(s(dbl(1)))):first(0,terms(3))`

$\rightarrow$ `recip(1):recip(s(s(s(s(dbl(0)))))):first(0,terms(3))`

$\rightarrow$ `recip(1):recip(4):first(0,terms(3))`

$\rightarrow$ `recip(1):recip(4):[]`

The expected result $[1, \frac{1}{4}]$ is obtained without any risk of nontermination.

Computing infinite normal forms

A TRS is infinitary normalizing if every (finite) term t admits a *strongly convergent sequence* (i.e., a rewrite sequence that, ultimately, reduces deeper and deeper redexes) starting from t and ending into a (possibly infinite) normal form.

A TRS is top-terminating if no infinitary reduction sequence performs infinitely many rewrites at topmost position (Dershowitz et al. [TCS'91]).

The following TRS $\mathcal{R}$:

$$\mathtt{f(a)} \to \mathtt{f(f(a))} \qquad\qquad \mathtt{f(a)} \to \mathtt{a}$$

is infinitary normalizing but not top-terminating:

$$\underline{\mathtt{f(a)}} \to \mathtt{f(}\underline{\mathtt{f(a)}}\mathtt{)} \to \underline{\mathtt{f(a)}} \to \cdots$$

Computing infinite normal forms

Top-terminating TRSs *only* admit strongly convergent sequences!

SEQUENCES

TRSS	Finite	Infinite
	Normalizing	Inf. normalizing
	Terminating	Top-terminating

Theorem Let $\mathcal{R}$ be a left-linear TRS and $\mu \in CM_{\mathcal{R}}$. If $\mathcal{R}$ is μ-terminating, then $\mathcal{R}$ is top-terminating.

Termination of canonical *CSR*

Termination of canonical *CSR* is an interesting property:

① For computing normal forms

② For proving top-termination

③ For approximating infinite normal forms

Termination of *CSR* by transformation

The μ-termination of a TRS $\mathcal{R}$ can be demonstrated by proving termination of a TRS $\mathcal{R}^\mu_\Theta$ for a given transformation Θ:

① Lucas [ICALP'96] $\mathcal{R}, \mu \mapsto \mathcal{R}^\mu_L$

② Zantema [RTA'97] $\mathcal{R}, \mu \mapsto \mathcal{R}^\mu_Z$

③ Ferreira and Ribeiro [RTA'99] $\mathcal{R}, \mu \mapsto \mathcal{R}^\mu_{FR}$

④ Giesl and Middeldorp [RTA'99] $\mathcal{R}, \mu \mapsto \mathcal{R}^\mu_{GM}$

All these transformations are incomplete (i.e., for all $\Theta \in \{L, Z, FR, GM\}$ there are $\mathcal{R}$ and μ such that $\mathcal{R}$ is μ-terminating but $\mathcal{R}^\mu_\Theta$ is not terminating).

Lucas' transformation [ICALP'96]

We remove all non-replacing subterms from the rules of the TRS $\mathcal{R}$.

Example For our guiding example, we obtain:

`sqr(0)`	$\to$	`0`	`0 + x`	$\to$	`x`
`sqr(s(x))`	$\to$	`s(sqr(x)+dbl(x))`	`s(x) + y`	$\to$	`s(x+y)`
`dbl(0)`	$\to$	`0`	`first(0,x)`	$\to$	`[]`
`dbl(s(x))`	$\to$	`s(s(dbl(x)))`	`first(s(x),:(y))`	$\to$	`:(y)`
`half(0)`	$\to$	`0`	`half(s(s(x)))`	$\to$	`s(half(x))`
`half(s(0))`	$\to$	`0`	`half(dbl(x))`	$\to$	`x`
`terms(n)`	$\to$	`:(recip(sqr(n)))`			

Terminating! (use an *rpo*)

Lucas' transformation [ICALP'96]

Let *CoCM*$_{\mathcal{R}}$ be the set of replacement maps $\mu \in$ *CM*$_{\mathcal{R}}$ satisfying that these removals do not yield rules with extra variables.

Theorem (Completeness) Let $\mathcal{R}$ be a left-linear TRS and $\mu \in$ *CoCM*$_{\mathcal{R}}$. If $\mathcal{R}$ is μ-terminating, then $\mathcal{R}_L^\mu$ is terminating.

Zantema's transformation [RTA'97]

The non-replacing subterms of the rules are marked:

`sqr(0)`	$\to$	`0`	`0 + x`	$\to$	`x`
`sqr(s(x))`	$\to$	`s(sqr(x)+dbl(x))`	`s(x) + y`	$\to$	`s(x+y)`
`dbl(0)`	$\to$	`0`	`first(0,x)`	$\to$	`[]`
`dbl(s(x))`	$\to$	`s(s(dbl(x)))`	`first(s(x),y:z)`	$\to$	`y:`$first'$`(x,`$activate$`(z))`
`half(0)`	$\to$	`0`	`half(s(s(x)))`	$\to$	`s(half(x))`
`half(s(0))`	$\to$	`0`	`half(dbl(x))`	$\to$	`x`

`terms(n)`	$\to$	`recip(sqr(n)):`$terms'$`(s(n))`	$activate(x)$	$\to$	x
$first(x,y)$	$\to$	$first'(x,y)$	$activate(first'(x,y))$	$\to$	$first(x,y)$
$terms(x)$	$\to$	$terms'(x)$	$activate(terms'(x))$	$\to$	$terms(x)$

Zantema's transformation [RTA'97]

The transformation remains incomplete for canonical replacement maps.

Ferreira and Ribeiro [RTA'99] describe a refinement of Zantema's transformation which is also incomplete for canonical replacement maps.

Giesl and Middeldorp's transformations [RTA'99]

The replacing subterms are marked (during computations): for all $l \to r \in R$ and $f \in \mathcal{F}$,

$$\begin{array}{rcl} \texttt{active}(l) & \to & \texttt{mark}(r) \\ \texttt{mark}(f(x_1,\ldots,x_k)) & \to & \texttt{active}(f([x_1]_f,\ldots,[x_k]_f)) \\ \texttt{active}(x) & \to & x \end{array}$$

where $[x_i]_f = \texttt{mark}(x_i)$ if $i \in \mu(f)$ and $[x_i]_f = x_i$ otherwise.

Theorem (Completeness) Let $\mathcal{R}$ be a left-linear TRS and $\mu \in$ *CM*$_{\mathcal{R}}$. If $\mathcal{R}$ is μ-terminating, then $\mathcal{R}^{\mu}_{GM}$ is terminating.

Giesl and Middeldorp's transformations [RTA'99]

Giesl and Middeldorp also proposed two refinements

$$\mathcal{R}, \mu \mapsto \mathcal{R}^{\mu}_{mGM} \quad \text{and} \quad \mathcal{R}, \mu \mapsto \mathcal{R}^{\mu}_{nGM}$$

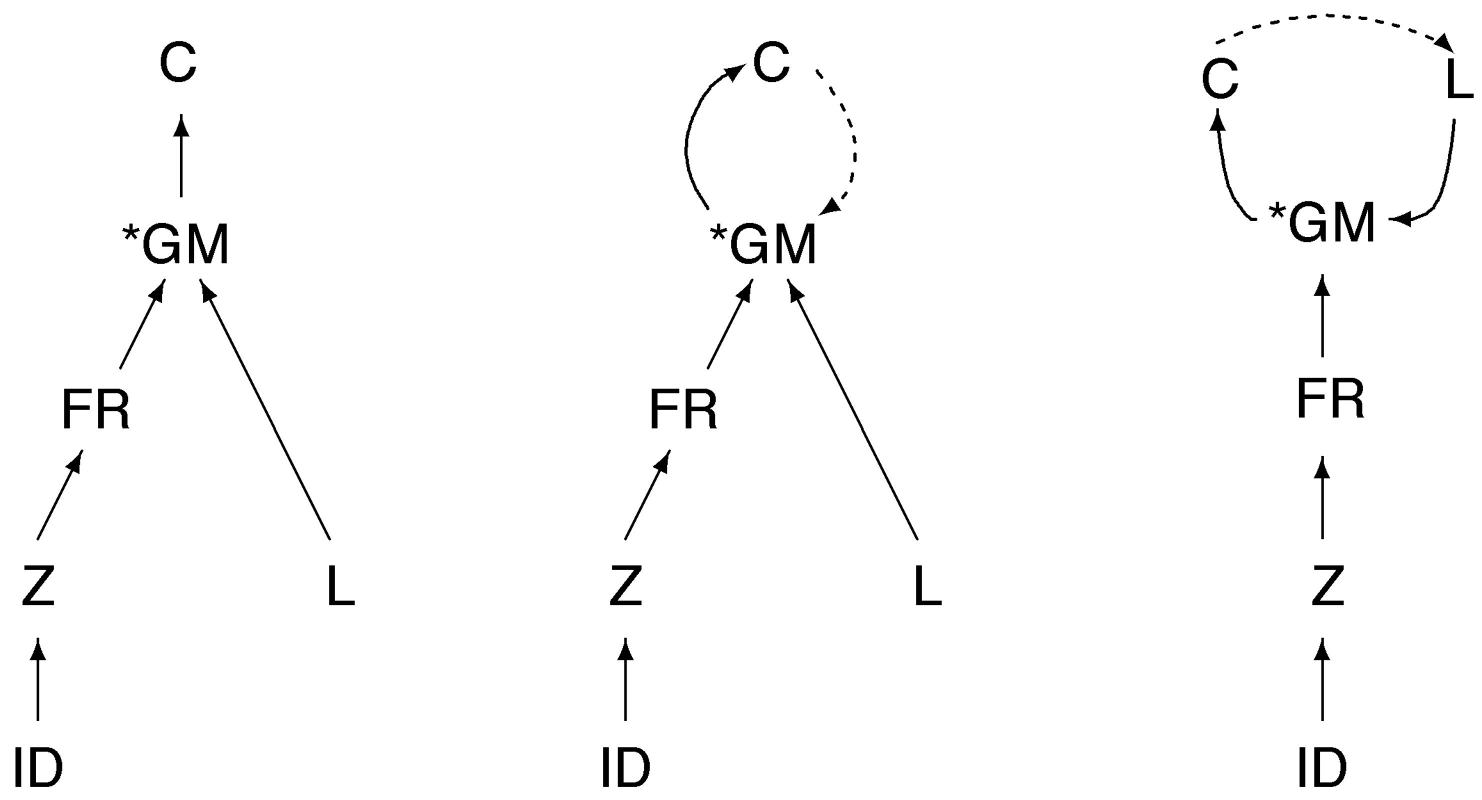
of this transformation.

They also define a complete transformation

$$\mathcal{R}, \mu \mapsto \mathcal{R}^{\mu}_{C}$$

(not described here).

Termination of *CSR*: transformations



$M_{\mathcal{R}}$ [RTA'99] $CM_{\mathcal{R}}$ $CoCM_{\mathcal{R}}$

Simple termination of the transformed systems

Simple termination covers the use of most usual automatizable orderings for proving termination of rewriting:

① Recursive path orderings

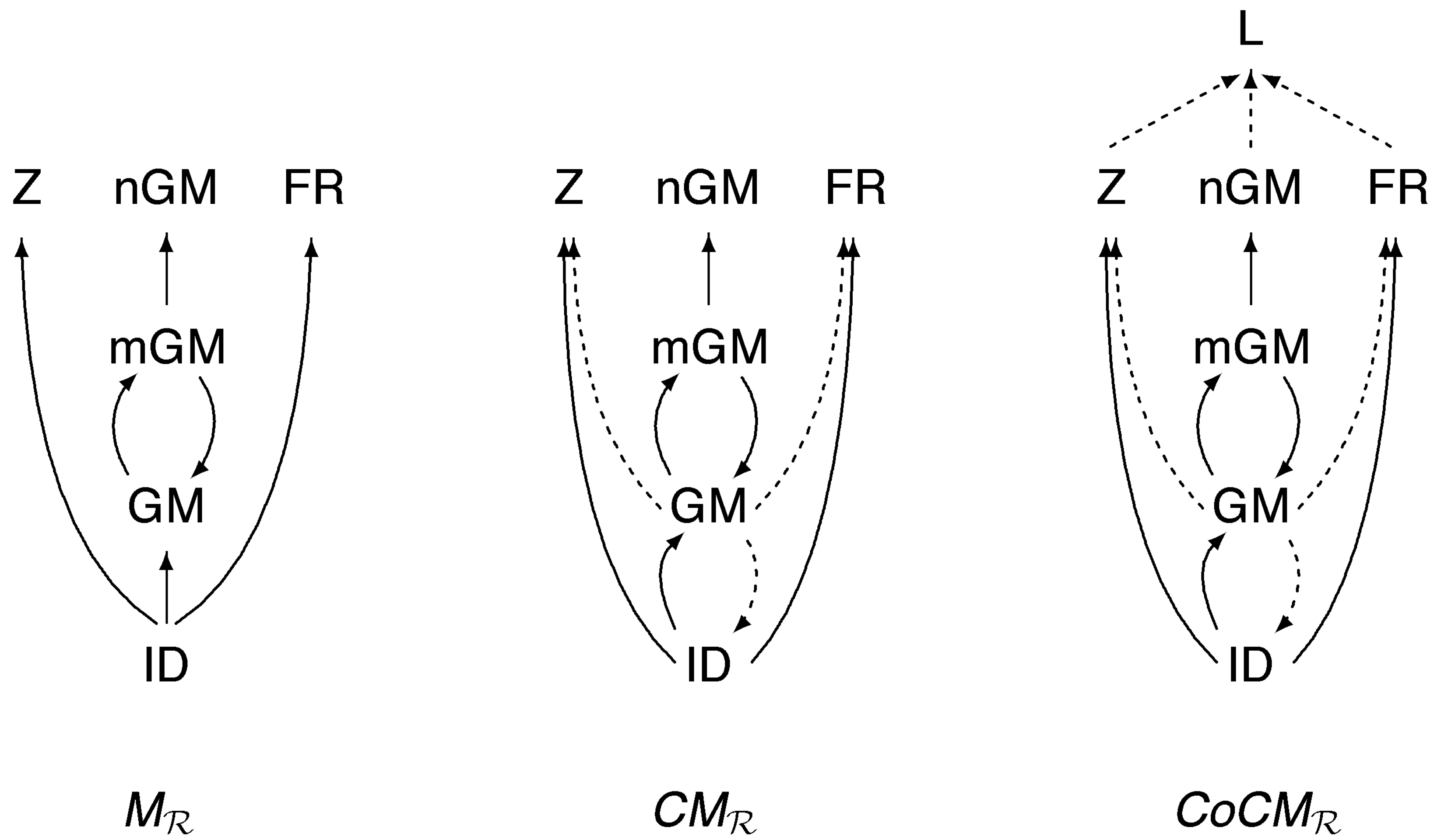
② Knuth-Bendix orderings

③ Polynomial orderings

An interesting problem:

can we use them for proving termination of the transformed systems $\mathcal{R}^{\mu}_{\Theta}$ *?*

Simple termination of the transformed systems



Termination of canonical *CSR* vs. termination

Termination of canonical *CSR* can be used:

① For computing normal forms (using $norm_\mu$)

② For proving top-termination

③ For approximating infinite normal forms

Hence, at least for left-linear TRSs (and the previous purposes), proving termination of canonical *CSR* could be priorized over proofs of termination.

What about the 'difficulty' of proving canonical termination?

Termination of canonical *CSR* vs. termination

		ID		L		Z		nGM	
Ref.	Example	Std	DG	Std	DG	Std	DG	Std	DG
5.	Non Simp.	N	0.06	0.03	0.00	0.04	0.00	N	0.15
7.	Dutch Flag	0.09	0.06	0.05	0.03	0.22	0.12	?	0.17
8.	Diff.	N	N	0.02	0.00	3.14	1.52	?	0.49
33.	Hydra	N	N	NC	NC	N	N	N	N
3.1.	Division v.1	N	0.43	NC	NC	?	0.89	?	N
3.5.	Remainder	N	?	NC	NC	?	?	?	?
3.7.	Logarithm	105.0	0.21	=ID	=ID	6469	0.21	?	?
3.10.	Min. sort	N	?	NC	NC	N	?	N	?

Experiments on termination vs. $\mu_{\mathcal{R}}^{can}$-termination with CiME 2.0

`http://www.dsic.upv.es/users/elp/slucas/experiments`

Conclusions

- Canonical *CSR* can be used for obtaining (infinite) normal forms
- Under certain conditions, Lucas', and Giesl and Middeldorp's transformations are complete for proving termination of canonical *CSR*.
- We have described a hierarchy of the transformations which is helpful for guiding their practical use.
- Termination of canonical *CSR* is a computational property which can be more interesting to analyze than standard termination. We provide (partial) evidence of this claim using some experimental results.

Future work

Comparing methods for proving termination of *CSR* is interesting for guiding their practical use. In this sense, some further work could be done:

- Very recently, some direct methods for proving termination of *CSR* have been described:
  ① CSRPO (Borralleras, Lucas, and Rubio [CADE'02])
  ② Polynomial orderings for *CSR* (Gramlich and Lucas [Draft'02])
  ③ CS Knuth-Bendix ordering (Borralleras [PhD'02])
  ④ Modular approach (Gramlich and Lucas [PPDP'02])

  These methods have been only partially related to transformational ones.

- Comparing the transformations w.r.t. particular techniques for proving termination (e.g., *rpo*, *kbo*, *poly*, Dep. pairs, etc.) is also interesting.