Efficient Strong Sequentiality Using Replacement Restrictions*

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Abstract. Huet and Lévy defined the (orthogonal) *strongly sequential* term rewriting systems, for which *index reduction*, i.e., reduction of redexes placed at special positions called (strong) *indices*, is optimal and normalizing. Despite the fact that Huet and Lévy give an algorithm to compute indices for the general case, there are many proposals to define subclasses of strongly sequential rewrite systems for which this can be done more efficiently. In this paper, we show that sometimes it is possible to enlarge such classes by only introducing fixed replacement restrictions, without forcing any sensible modification of the corresponding index reduction strategy.

Keywords: functional programming, neededness, replacement restrictions, term rewriting.

1 Introduction

For orthogonal term rewriting systems (TRSs), Huet and Lévy gave a formal basis for the definition of efficient *sequential* strategies, i.e., reduction sequences in which only one needed redex is reduced in each step [2, 4]. The occurrences which is worthy to further explore to normalize a term are called (sequential) indices. Reduction of redexes placed at sequential indices (index reduction) is normalizing. In general, sequential indices are not computable, but Huet and Lévy define a class of orthogonal TRSs, the strongly sequential TRSs, for which this can be done. It is also decidable whether a TRS is strongly sequential.

The general algorithm to perform index reduction is rather complex, and several approximations have been developed in order to obtain more efficient, but less general methods [8, 9, 10]. A remarkable example are the left-normal TRSs (LN), a subclass of strongly sequential TRSs (SS), for which the leftmost-outermost strategy, an easily computable strategy, is indeed index reduction [8].

Context-sensitive rewriting (*csr*) [5] is rewriting with fixed restrictions on replacements which are completely determined by the symbols of the signature:

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a mapping $\mu : \Sigma \to \mathcal{P}(\mathbb{N})$ called the *replacement map* indicates the argument positions which can be replaced for each symbol f. The arguments of f indexed by $\mu(f)$ can be rewritten, and the others do not. The restrictions are lifted in the natural way from arguments of functions to arbitrary occurrences of terms.

Example 1.1 *Let us consider the TRS $\mathcal{R}$:*

$\mathtt{zero} \to \mathtt{0}$ $\qquad\qquad\qquad\qquad$ $\mathtt{first(0,x)} \to []$

$\mathtt{from(x)} \to \mathtt{x} :: \mathtt{from(s(x))}$ $\qquad$ $\mathtt{first(s(x), y :: z)} \to \mathtt{y} :: \mathtt{first(x,z)}$

We define $\mu(\mathtt{s}) = \mu(::) = \mu(\mathtt{from}) = \emptyset$ and $\mu(\mathtt{first}) = \{1,2\}$. We can perform the following context-sensitive derivation:

$\mathtt{first}(\underline{\mathtt{zero}}, \mathtt{from(0)}) \to \underline{\mathtt{first(0,from(0))}} \to []$,

but the infinite derivation

$\mathtt{first(zero}, \underline{\mathtt{from(0)}}) \to \mathtt{first(zero}, \mathtt{0} :: \underline{\mathtt{from(s(0))}}) \to \cdots$

is avoided due to the restriction $\mu(::) = \emptyset$.

In a computational system dealing with a certain subclass CL of SS, we have an index reduction strategy $\mathbb{F}$ which is normalizing for the TRSs in CL. In this paper we give a general method to incorporate replacement restrictions into the TRSs of any class $CL \subseteq SS$. This can lead to an effective extension of the considered class CL. We show how to compute the corresponding index reduction (restricted) strategy $\mathbb{F}^\mu$ for each TRS by using the information in $\mathbb{F}$.

In Section 2, we give some preliminary definitions. In Section 3, we introduce *csr*. Section 4 reviews the main concepts and results about strong sequentiality. Section 5 studies the general relations between replacement restrictions and strong sequentiality. Section 6 explains how to adapt the strategies.

2 Preliminaries

We denote as V a countable set of variables and Σ denotes a signature: a set of function symbols $\{\mathtt{f}, \mathtt{g}, \ldots\}$, each with a fixed arity given by a function $ar : \Sigma \to \mathbb{N}$. By $\mathcal{T}(\Sigma, V)$, we denote the set of terms. Denote as $\tilde{t}$ a k-tuple $t_1, \ldots, t_k$ of terms where k will be clarified by the context. $Var(t)$ is the set of variables in t.

Terms are viewed as labelled trees in the usual way. Occurrences $u, v, \ldots$ are represented by chains of positive natural numbers used to address subterms of t. Occurrences are ordered by the standard prefix ordering: $u \leq v$ iff $\exists v'$ s.t. $v = u.v'$. Also, $u \parallel v$ means $u \not\leq v$ and $v \not\leq u$. We also use the *total* ordering $\leq_L$ given by: $\epsilon \leq_L u$ and $i.u \leq_L j.vv \Leftrightarrow i < j \vee (i = j \wedge u \leq_L v)$. Denote as $O(t)$ the set of occurrences of a term t. The occurrences of non-variable symbols in t are $O_\Sigma(t)$, and $O_V(t)$ are the variable occurrences. The subterm at occurrence u of t is $t|_u$ and $t[s]_u$ is the term t with the subterm at the occurrence u replaced with s. The symbol labelling the root of t is $root(t)$ and $\Sigma(t) = \{f \in \Sigma \mid \exists u \in O(t).root(t|_u) = f\}$ is the set of symbols from Σ appearing in t. We refer to any term C, which is the same as t everywhere except below u, i.e. there exists a term s such that $C[s]_u = t$ (or just $C[s]$), as the *context* within the replacement occurs.

A rewrite rule is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\Sigma, V)$, $l \notin V$ and $Var(r) \subseteq Var(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A TRS is a pair $\mathcal{R} = (\Sigma, R)$ where R is a finite set of rewrite rules. Denote as $L(\mathcal{R})$ the *lhs*'s of $\mathcal{R}$. A TRS is left-linear if there are no multiple occurrences of a single variable in the *lhs*'s of the rules. Two rules $l \to r$ and $l' \to r'$ *overlap*, if there is a non-variable occurrence $u \in O(l)$, a substitution σ such that $\sigma(l|_u) = \sigma(l')$. A TRS is non-ambiguous, if there are no overlapping *lhs*'s (trivial overlap in the same *lhs* is not considered). A left-linear, non-ambiguous TRS is called orthogonal. An instance $\sigma(l)$ of a *lhs* l of a rule is a redex. The set of redex occurrences in t is $O_{\mathcal{R}}(t) = \{u \in O(t) \mid \exists l \in L(\mathcal{R}) \ : \ t|_u = \sigma(l)\}$. We say that t rewrites to s (at the occurrence u), written $t \xrightarrow{u}_{\mathcal{R}} s$ (or just $t \to s$), if $t|_u = \sigma(l)$ and $s = t[\sigma(r)]_u$, for some rule $l \to r \in R$, $u \in O(t)$ and substitution σ. A term t is in head-normal form (*hnf*) if there is no derivation $t = t_1 \to t_2 \to \cdots$ which reduces the root of a term t_i $i \geq 1$.

$\mathbb{N}_k^+$ is an initial segment $\{1, 2, \ldots k\}$ of positive natural numbers, where $\mathbb{N}_0^+ = \emptyset$. $\mathcal{P}(\mathbb{N})$ is the powerset of natural numbers. We denote as $\mathcal{P}(\mathbb{N}^*_{>0})$ the powerset of chains of positive numbers.

3 Context-sensitive rewriting

Given a signature Σ, a mapping $\mu : \Sigma \to \mathcal{P}(\mathbb{N})$ is a *replacement map* (or Σ-map) if for all $f \in \Sigma$. $\mu(f) \subseteq \mathbb{N}^+_{ar(f)}$. Hence, μ determines the *argument* positions which can be reduced for each symbol in the signature [5]. We define an ordering $\sqsubseteq$ on M_Σ, the set of all Σ-maps: $\mu \sqsubseteq \mu'$ if for all $f \in \Sigma$, $\mu(f) \subseteq \mu'(f)$. Then, $(M_\Sigma, \sqsubseteq, \mu_\bot, \mu_\top, \sqcup, \sqcap)$ is a lattice: for all $f \in \Sigma$, $\mu_\bot(f) = \emptyset$, $\mu_\top(f) = \mathbb{N}^+_{ar(f)}$, $(\mu \sqcup \mu')(f) = \mu(f) \cup \mu'(f)$ and $(\mu \sqcap \mu')(f) = \mu(f) \cap \mu'(f)$. Hence, $\mu \sqsubseteq \mu'$ means that μ considers less positions than μ' for reduction.

The set of μ-*replacing* occurrences $O^\mu(t)$ is given by: $O^\mu(x) = \{\epsilon\}$, if $x \in V$ and $O^\mu(f(\tilde{t})) = \{\epsilon\} \cup \bigcup_{i \in \mu(f)} i.O^\mu(t_i)$. The *non-replacing* occurrences are $\widetilde{O^\mu}(t) = O(t) \backslash O^\mu(t)$. In *csr*, we rewrite *replacing* occurrences: t μ-rewrites to s, written $t \hookrightarrow_{\mathcal{R}(\mu)} s$ (or simply $\hookrightarrow$), if $t \xrightarrow{u}_{\mathcal{R}} s$ and $u \in O^\mu(t)$. The set of *replacing redexes* is $O^\mu_{\mathcal{R}}(t) = O_{\mathcal{R}}(t) \cap O^\mu(t)$ and t is a μ-normal form, if $O^\mu_{\mathcal{R}}(t) = \emptyset$. The set of replacing variables is $Var^\mu(t) = \{x \in Var(t) \mid \exists u \in O(t).\ t|_u = x\}$.

A term $t \in \mathcal{T}(\Sigma, V)$ is μ-*compatible* if every non variable occurrence of t is a replacing occurrence: $O_\Sigma(t) \subseteq O^\mu(t)$. Given a term t, the *minimum* $\Sigma(t)$-map μ_t ensuring that t is compatible is: if $t = x$, then $\mu_t(f) = \emptyset$ for all $f \in \Sigma$. If $t = f(t_1, \ldots, t_k)$, then $\mu_t = \mu_t^\epsilon \sqcup \mu_{t_1} \sqcup \cdots \sqcup \mu_{t_k}$, where μ_t^ϵ is $\mu_t^\epsilon(f) = \{i \mid t_i \notin V\}$ and $\mu_t^\epsilon(g) = \emptyset$ for all $g \neq f$.

Example 3.1 *If* $t = \mathtt{first}(\mathtt{0}, \mathtt{x})$, *then* $\mu_t(\mathtt{first}) = \{1\}$. *Since* $\mu_{\mathtt{s}(\mathtt{x})}(\mathtt{s}) = \emptyset$ *and* $\mu_{\mathtt{y}::\mathtt{z}}(::) = \emptyset$, *for* $t' = \mathtt{first}(\mathtt{s}(\mathtt{x}), \mathtt{y} :: \mathtt{z})$, *we get* $\mu_{t'}(\mathtt{s}) = \mu_{t'}(::) = \emptyset$ *and* $\mu_{t'}(\mathtt{first}) = \{1, 2\}$.

The *canonical replacement map* for $\mathcal{R}$, $\mu^{com}_{\mathcal{R}} = \sqcup_{l \in L(\mathcal{R})} \mu_l$, is the minimum replacement map which makes the *lhs*'s of $\mathcal{R}$ compatible terms. For instance, μ in Example 1.1 is the canonical replacement map for the TRS $\mathcal{R}$ in the example.

4 Strong sequentiality

Indices point to unexplored parts of a term t which is worthy to further develop to normalize t. To represent *unknown* parts of t, a new constant symbol Ω is introduced. Terms in $\mathcal{T}(\Sigma \cup \{\Omega\}, V)$ are said Ω-terms. Denote as $O_\Omega(t)$ the set of occurrences of Ω in t: $O_\Omega(t) = \{u \in O(t) \mid t|_u = \Omega\}$. An ordering $\leq$ on Ω-terms is given: $\Omega \leq t$ for all t, $x \leq x$ if $x \in V$, and $f(\tilde{t}) \leq f(\tilde{s})$ if $t_i \leq s_i$ for all $1 \leq i \leq ar(f)$. Thus $t \leq s$ means 't is less defined than s'. An Ω-normal form t is s.t. $O_\mathcal{R}(t) = \emptyset$ and $O_\Omega(t) \neq \emptyset$. A redex scheme is a *lhs* in $L(\mathcal{R})$ where all variables are replaced by Ω. Let $L_\Omega(\mathcal{R})$ be the set of *redex scheme* of $\mathcal{R}$.

To calculate an index we use a function ω defined by means of a reduction relation $\to_\Omega$ [4]: $C[t] \to_\Omega C[\Omega]$ if $t \neq \Omega$ and there is $l \in L_\Omega(\mathcal{R})$ such that $t \uparrow l$, i.e., there exists $s \in \mathcal{T}(\Sigma \cup \{\Omega\}, V)$, such that $t \leq s$ and $l \leq s$. Then, $\to_\Omega$ is confluent and terminating (see [2, 4]). Let $\omega(t)$ be the $\to_\Omega$-normal form of t.

Definition 4.1 ([4]) *Let $t \in \mathcal{T}(\Sigma \cup \{\Omega\}, V)$ and $u \in O_\Omega(t)$. Let $\bullet$ be a fresh constant symbol, and $t' = t[\bullet]_u$. Then u is an* index *of t, iff $\omega(t')|_u = \bullet$. Let us denote as $\mathcal{I}(t)$ the set of indices of t.*

A term t is *rigid* if $\omega(t) = t$, *soft* if $\omega(t) = \Omega$ [3] and a *strong hnf* if $\omega(t) \neq \Omega$.

Proposition 4.2 *If t is a strong* hnf, *then t is a* hnf.

Definition 4.3 ([3]) *An orthogonal TRS is strongly sequential if every Ω-normal form has an index.*

Any strategy which always reduces redexes pointed by indices is *index reduction*.

Theorem 4.4 ([2]) *Index reduction is normalizing for orthogonal, strongly sequential TRSs.*

5 Strong sequentiality and replacement restrictions

In this section, we analyze the relation between replacement restrictions given by a replacement map μ and indices of terms w.r.t. a TRS $\mathcal{R}$. We consider *replacing* indices, $\mathcal{I}^\mu(t) = \mathcal{I}(t) \cap O^\mu(t)$, and *non-replacing* ones: $\widetilde{\mathcal{I}^\mu}(t) = \mathcal{I}(t) \cap \widetilde{O^\mu}(t)$.

Proposition 5.1 *Let $\mathcal{R}$ be a TRS, and μ be such that $\mu_\mathcal{R}^{com} \sqsubseteq \mu$. Let $t \in \mathcal{T}(\Sigma \cup \{\Omega\}, V)$ be a soft term. Then $\widetilde{\mathcal{I}^\mu}(t) = \emptyset$.*

From Proposition 4.2, if t is not a *hnf*, then t is not a strong *hnf*. Hence t is soft. Thus, Proposition 5.1 entails that, in reductions leading to a *hnf*, *only replacing redex occurrences must be considered*, since there are no non-replacing indices.

Theorem 5.2 *Let $\mathcal{R}$ be an orthogonal, strongly sequential TRS, and μ be such that $\mu_\mathcal{R}^{com} \sqsubseteq \mu$. Let $t \in \mathcal{T}(\Sigma \cup \{\Omega\}, V)$ be an Ω-normal form, such that $O_\Omega^\mu(t) \neq \emptyset$. Then $\mathcal{I}^\mu(t) \neq \emptyset$.*

This result means that an index reduction strategy can be refined by including replacement restrictions without damage its index reduction nature.

5.1 The contractive transformation

Now, we connect strong sequentiality of a TRS $\mathcal{R}$ and strong sequentiality of a TRS $\mathcal{R}^\mu$ obtained by means of a transformation called *μ-contraction* which we have introduced in a previous work [7]: The *μ-contracted signature* Σ^μ is defined as follows: $f_\mu \in \Sigma^\mu \wedge ar(f_\mu) = |\mu(f)| \Leftrightarrow f \in \Sigma$. Terms are related by means of a *μ-contracting function* $\tau_\mu : \mathcal{T}(\Sigma, V) \to \mathcal{T}(\Sigma^\mu, V)$. This function *drops* the non-replacing immediate subterms of a term t and constructs a 'μ-contracted' term by joining the (also transformed) replacing subterms below the corresponding operation of the μ-contracted signature: $\tau_\mu(x) = x$ if $x \in V$, and $\tau_\mu(f(t_1, \ldots, t_k)) = f_\mu(\tau_\mu(t_{i_1}), \ldots, \tau_\mu(t_{i_p}))$ if $\mu(f) = \{i_1, \ldots, i_p\}$. The i-th immediate (replacing) subterm of f in $t = f(\tilde{t})$ becomes the $|\mathbb{N}_i^+ \cap \mu(\mathrm{f})|$-th immediate subterm of f_μ in $\tau_\mu(t)$.

Example 5.3 *Let $\mathcal{R}$ and $\mu = \mu_\mathcal{R}^{com}$ as given in Example 1.1.*

Then $t = \mathtt{first}(\mathtt{s}(\mathtt{x}), \mathtt{y} :: \mathtt{z})$ contracts to $\tau_\mu(t) = \mathtt{first}_\mu(\mathtt{s}_\mu, ::_\mu)$ and $t' = \mathtt{y} :: \mathtt{first}(\mathtt{x}, \mathtt{z})$ contracts to $\tau_\mu(t') = ::_\mu$.

Since τ_μ is surjective, we are allowed to consider any $s \in \mathcal{T}(\Sigma^\mu, V)$ as the image by τ_μ of some $t \in \mathcal{T}(\Sigma, V)$. We write $\tau_\mu(t)$ instead of s when $s = \tau_\mu(t)$.

Replacing occurrences of t and occurrences of $\tau_\mu(t)$ are related by $\nu_{\mu,t} : O^\mu(t) \to O(\tau_\mu(t))$: $\nu_{\mu,t}(\epsilon) = \epsilon$, and $\nu_{\mu,f(\tilde{t})}(i.u) = |\mathbb{N}_i^+ \cap \mu(f)|.\nu_{\mu,t_i}(u)$ if $i \in \mu(f)$. We write ν_t instead of $\nu_{\mu,t}$ when μ is clear from the context. Note that $\nu_{\mu,t}$ is a bijection, and $\nu_{\mu,t}(O^\mu(t)) = O(\tau_\mu(t))$.

We use τ_μ to connect TRSs. Let $\mathcal{R} = (\Sigma, R)$ be a TRS. The μ-contraction of $\mathcal{R}$ is $\mathcal{R}^\mu = (\Sigma^\mu, R^\mu)$, where $R^\mu = \{\tau_\mu(l) \to \tau_\mu(r) \mid l \to r \in R\}$. We indicate as $\rho_\mu(l \to r)$ the rule of $\mathcal{R}^\mu$ which corresponds to a rule $l \to r$ of $\mathcal{R}$. ρ_μ is surjective.

Example 5.4 *By considering $\mathcal{R}$ and μ as in Example 1.1, $\mathcal{R}^\mu$ is:*

$\mathtt{zero}_\mu \to \mathtt{0}_\mu$	$\mathtt{first}_\mu(\mathtt{0}_\mu, \mathtt{x}) \to [\,]_\mu$
$\mathtt{from}_\mu \to ::_\mu$	$\mathtt{first}_\mu(\mathtt{s}_\mu, ::_\mu) \to ::_\mu$

The definition of $\mathcal{R}^\mu$ only has complete sense if $Var^\mu(r) \subseteq Var^\mu(l)$ for each $l \to r \in R$. In this way, $\mathcal{R}^\mu$ has no extra variables. For instance, in Example 5.4, we have obtained a valid TRS. However, if we define $\mu(::) = \{1\}$, the new contraction does not yield a TRS, because we obtain extra variables. Nevertheless, concerning the results in this paper, we are independent from this circumstance.

5.2 Inducing strong sequentiality

In the following, we use $\mathcal{J}(t)$ to denote indices obtained using $L_\Omega(\mathcal{R}^\mu)$.

Proposition 5.5 *Let $\mathcal{R}$ be a TRS and μ be such that $\mu_\mathcal{R}^{com} \sqsubseteq \mu$. Let t be an Ω-term. Then $u \in \mathcal{I}^\mu(t)$ iff $\nu_t(u) \in \mathcal{J}(\tau_\mu(t))$, i.e., $\nu_t(\mathcal{I}^\mu(t)) = \mathcal{J}(\tau_\mu(t))$.*

Strong sequentiality is independent from *rhs*'s of rules. Since τ_μ can introduce extra variables in the *rhs*'s, in order to formulate our results, we say that TRSs $\mathcal{R} = (\Sigma, R)$ and $\mathcal{R}' = (\Sigma', R')$ are *left-equivalent* ($\mathcal{R} \sim_l \mathcal{R}'$) if they have the same *lhs*'s, i.e., $R = \{l_i \to r_i \mid 1 \le i \le n\}$ and $R' = \{l_i \to r'_i \mid 1 \le i \le n\}$ [4].

Theorem 5.6 *Let $\mathcal{R}$ be an orthogonal TRS. Let μ be such that $\mu_{\mathcal{R}}^{com} \sqsubseteq \mu$ and $\mathcal{R}^\mu$ is left equivalent to a strongly sequential TRS. Then $\mathcal{R}$ is strongly sequential.*

To illustrate Theorem 5.6, we recall the notion of left-normal TRS [8, 10]: $\mathcal{R}$ is left-normal if, for all $l \in L(\mathcal{R})$, the function symbols *precede* the variables according to $\leq_L$. Thus $\mathcal{R}$ in Example 1.1, is *not* left-normal. No easy syntactic characterization of strong sequentiality for $\mathcal{R}$ is available. However, $\mathcal{R}^\mu$, where $\mu = \mu_{\mathcal{R}}^{com}$, in Example 5.4 is left-normal, hence strongly sequential. Therefore, Theorem 5.6 entails strong sequentiality of $\mathcal{R}$. We give a further example.

Example 5.7 *Let us consider the TRS $\mathcal{R}$ to manipulate prime numbers.*

$\mathtt{from(x)} \to \mathtt{x} :: \mathtt{from(s(x))}$
$\mathtt{primes} \to \mathtt{sieve(from(s(s(0))))}$
$\mathtt{sieve(x} :: \mathtt{y)} \to \mathtt{x} :: \mathtt{sieve(filter(x,y))}$
$\mathtt{filter(x,y} :: \mathtt{z)} \to \mathtt{if(x|y,filter(x,z),y} :: \mathtt{filter(x,z))}$
$\mathtt{sel(0,x} :: \mathtt{y)} \to \mathtt{x}$ $\quad$ $\mathtt{if(true,x,y)} \to \mathtt{x}$
$\mathtt{sel(s(x),y} :: \mathtt{z)} \to \mathtt{sel(x,z)}$ $\quad$ $\mathtt{if(false,x,y)} \to \mathtt{y}$

Assume that $|$ is a built-in predicate and $x|y$ means x divides *y. If we let $\mu = \mu_{\mathcal{R}}^{com}$, then $\mu(\mathtt{s}) = \mu(::) = \mu(\mathtt{from}) = \emptyset$, $\mu(\mathtt{sieve}) = \mu(\mathtt{if}) = \{1\}$, $\mu(\mathtt{filter}) = \{2\}$, $\mu(\mathtt{sel}) = \{1,2\}$. $\mathcal{R}$ is* not *left-normal. Then $\mathcal{R}^\mu$ is:*

$\mathtt{from}_\mu \to ::_\mu$
$\mathtt{primes}_\mu \to \mathtt{sieve}_\mu(\mathtt{from}_\mu)$
$\mathtt{sieve}_\mu(::_\mu) \to ::_\mu$
$\mathtt{filter}_\mu(::_\mu) \to \mathtt{if}_\mu(|_\mu)$
$\mathtt{sel}_\mu(\mathtt{0}_\mu, ::_\mu) \to \mathtt{x}$ $\quad$ $\mathtt{if}_\mu(\mathtt{true}_\mu) \to \mathtt{x}$
$\mathtt{sel}_\mu(\mathtt{s}_\mu, ::_\mu) \to \mathtt{sel}_\mu(\mathtt{x},\mathtt{z})$ $\quad$ $\mathtt{if}_\mu(\mathtt{false}_\mu) \to \mathtt{y}$

$\mathcal{R}^\mu$ is not a TRS (it has extra variables), but it is immediate to have a left-equivalent TRS $\mathcal{S} \sim_l \mathcal{R}^\mu$. $\mathcal{S}$ must be a ground *TRS (i.e., $l, r \in \mathcal{T}(\Sigma, \emptyset)$ for all rule $l \to r$ of $\mathcal{S}$), since the lhs's have no variable. Ground TRSs are left-normal, hence strongly sequential. Thus, $\mathcal{R}$ also is.*

Theorem 5.8 *Let $\mathcal{R}$ be an orthogonal, strongly sequential TRS. Let μ be such that $\mu_{\mathcal{R}}^{com} \sqsubseteq \mu$. Then, $\mathcal{R}^\mu$ is left equivalent to a strongly sequential TRS.*

Hence, Theorems 5.6 and 5.8 amount to saying that proving strong sequentiality in a TRS $\mathcal{R}$ or its μ-contraction $\mathcal{R}^\mu$ is completely equivalent, whenever $\mu_{\mathcal{R}}^{com} \sqsubseteq \mu$.

6 Induced strategies

We have shown that it is possible to conclude strong sequentiality of $\mathcal{R}$ from strong sequentiality of (a TRS left-equivalent to) $\mathcal{R}^\mu$. In this section, we show how to obtain an index reduction strategy for $\mathcal{R}$ by using an index reduction strategy for $\mathcal{R}^\mu$.

A rewriting strategy for a TRS $\mathcal{R} = (\Sigma, R)$ is a function $\mathbb{F}_{\mathcal{R}} : \mathcal{T}(\Sigma, V) \to \mathcal{T}(\Sigma, V)$ such that $\mathbb{F}_{\mathcal{R}}(t) = t$ if $O_{\mathcal{R}}(t) = \emptyset$, i.e, t is a normal form, and $t \to_{\mathcal{R}}^{+} \mathbb{F}_{\mathcal{R}}(t)$ otherwise ($t \to_{\mathcal{R}} \mathbb{F}_{\mathcal{R}}(t)$ for sequential strategies) [3]. We drop the suffix

$\mathcal{R}$ when no confusion arises. Also, we split $\mathbb{F}$ into components $\mathbb{F} = (\mathbb{P}, \mathbb{R})$. The component $\mathbb{P} : \mathcal{T}(\Sigma, V) \to \mathcal{P}(\mathbb{N}^*_{>0})$ selects the redex occurrences to be reduced (a singleton for sequential strategies), and $\mathbb{R}$ selects the rules which apply (see [1]). Since we only deal with orthogonal TRSs, the rule which applies to a redex is uniquely determined by the redex itself. Then, for a given $\mathbb{P}$, we get $\mathbb{R}$ fixed. Hence, we do not pay more attention to $\mathbb{R}$. Then, $\mathbb{P}$ and $\mathbb{F}$ are related by:

$\mathbb{P}(t) = \emptyset$ iff $O_{\mathcal{R}}(t) = \emptyset$.

$\mathbb{P}(t) = \{u_1, \ldots, u_n\} \subseteq O_{\mathcal{R}}(t)$ iff $u_i \parallel u_j$ for $1 \leq i < j \leq n$, $t|_{u_i} = \sigma(l_i)$ for a single $l_i \to r_i \in R$, $1 \leq i \leq n$, and $\mathbb{F}(t) = t[\sigma(r_1)]_{u_1} \cdots [\sigma(r_n)]_{u_n}$.

The notion of context-sensitive strategy or *μ-strategy* $\mathbb{F}_{\mathcal{R}(\mu)}$ is analogous, but it uses $\hookrightarrow_{\mathcal{R}(\mu)}$ instead of $\to_{\mathcal{R}}$ and $O^{\mu}_{\mathcal{R}}$ instead of $O_{\mathcal{R}}$ (see [6]). However, note that, from μ-strategies, we only can expect to μ-normalize terms, rather than completely normalize them[2]. Now, we introduce the notion of *induced* μ-strategy.

Definition 6.1 *Let $\mathcal{R} = (\Sigma, R)$ be an orthogonal TRS and μ be such that $\mu^{com}_{\mathcal{R}} \sqsubseteq \mu$. Let $\mathbb{F} = (\mathbb{P}, \mathbb{R})$ be a rewriting strategy for $\mathcal{S} \sim_l \mathcal{R}^{\mu}$. The μ-rewriting strategy $\mathbb{F}^{\mu} = (\mathbb{P}^{\mu}, \mathbb{R}^{\mu})$* induced *by $\mathbb{F}$ is given by $\mathbb{P}^{\mu}(t) = \nu_t^{-1}(\mathbb{P}(\tau_{\mu}(t)))$ for all $t \in \mathcal{T}(\Sigma, V)$. Since whenever $\mu^{com}_{\mathcal{R}} \sqsubseteq \mu$ the μ-contraction preserves orthogonality, $\mathbb{R}^{\mu}$ is again fixed by $\mathbb{P}^{\mu}$.*

The definition of $\mathbb{P}^{\mu}$ should be obvious after Proposition 5.5. Now we prove that $\mathbb{F}^{\mu}$ in Definition 6.1 holds the expected property, i.e., it is an index reduction μ-strategy for the TRS $\mathcal{R}$, whenever $\mathbb{F}$ is index reduction for (a TRS left-equivalent to) $\mathcal{R}^{\mu}$.

Theorem 6.2 *Let $\mathcal{S}$ be an orthogonal strongly sequential TRS. Let $\mathbb{F}$ be an index reduction strategy for $\mathcal{S}$. Let $\mathcal{R}$ be a TRS and μ be such that $\mu^{com}_{\mathcal{R}} \sqsubseteq \mu$ and $\mathcal{R}^{\mu} \sim_l \mathcal{S}$. Then, $\mathbb{F}^{\mu}$ is an index reduction μ-strategy for $\mathcal{R}$.*

Sometimes, $\mathbb{P}$ can be further decomposed as follows: $\mathbb{P} = \widehat{\mathbb{P}} \circ O_{\mathcal{R}}$ where $\widehat{\mathbb{P}} : \mathcal{P}(\mathbb{N}^*_{>0}) \to \mathcal{P}(\mathbb{N}^*_{>0})$ selects a set of occurrences from a non empty set of redex occurrences ($\widehat{\mathbb{P}}(\emptyset) = \emptyset$), and $O_{\mathcal{R}} : \mathcal{T}(\Sigma, V) \to \mathcal{P}(\mathbb{N}^*_{>0})$ gives the redex occurrences of t. We also require $\widehat{\mathbb{P}}(U) \subseteq U$ for all $U \in \mathcal{P}(\mathbb{N}^*_{>0})$. Some examples [3]: *Leftmost-outermost* ($\mathbb{F}_{lo}$): $\mathbb{P}_{lo}(t) = \{min_{\leq_L}(minimal_{\leq}(O_{\mathcal{R}}(t)))\} = \{min_{\leq_L}(O_{\mathcal{R}}(t))\}$. *Parallel-outermost* ($\mathbb{F}_{po}$): $\mathbb{P}_{po}(t) = minimal_{\leq}(O_{\mathcal{R}}(t))$. Note that in these strategies, the question of *which* rule should be applied to a selected redex is left open: In orthogonal TRSs the redex fixes the rule.

The calculus of the $\mathbb{P}^{\mu}$ component of an induced strategy (Definition 6.1), for a given term t, proceeds by using τ_{μ} for moving to $\mathcal{R}^{\mu}$, next using the $\mathbb{P}$ component of the original strategy and coming back to $\mathcal{R}$ by means of ν_t^{-1}. The decomposition $\mathbb{P} = \widehat{\mathbb{P}} \circ O_{\mathcal{R}}$ can lead to a simplification of this procedure.

[2] Nevertheless, it is easy to use μ-normalizing strategies to define normalizing procedures [6].

Proposition 6.3 *Let $\mathcal{R}$ be an orthogonal TRS, and μ be s.t. $\mu_{\mathcal{R}}^{com} \sqsubseteq \mu$. Let $\mathbb{F} = (\mathbb{P}, \mathbb{R})$ be a rewriting strategy for $\mathcal{S} \sim_l \mathcal{R}^\mu$ and $\mathbb{F}^\mu = (\mathbb{P}^\mu, \mathbb{R}^\mu)$ be the induced μ-strategy for $\mathcal{R}$. Let $\mathbb{P}$ be such that $\mathbb{P} = \widehat{\mathbb{P}} \circ O_{\mathcal{R}^\mu}$. If, for all $t \in \mathcal{T}(\Sigma, V)$ and $U \subseteq O^\mu(t)$, we have $\nu_t(\widehat{\mathbb{P}}(U)) = \widehat{\mathbb{P}}(\nu_t(U))$, then $\mathbb{P}^\mu(t) = \widehat{\mathbb{P}}(O_{\mathcal{R}}^\mu(t))$.*

The result on normalization by means of $\mathbb{F}_{lo}$ can be given by showing that an orthogonal, left-normal TRS is strongly sequential and the leftmost-outermost redex occurrence $min_{\leq_L}(O_{\mathcal{R}}(t))$ of a term t is an index (O'Donnell, [8]).

We show that $min_{\leq_L}$ commutes with ν. This allows us to give a simpler expression of the induced leftmost-outermost strategy $\mathbb{F}_{lo}^\mu$.

Proposition 6.4 *Let $t \in \mathcal{T}(\Sigma, V)$. $\forall U \subseteq O^\mu(t)$, $\nu_t(min_{\leq_L}(U)) = min_{\leq_L}(\nu_t(U))$.*

Therefore, from Proposition 6.4 and Proposition 6.3, the leftmost-outermost strategy $\mathbb{F}_{lo}$ induces a μ-strategy $\mathbb{F}_{lo}^\mu$ for a given $\mathcal{R}$ and μ such that $\mathcal{R}^\mu \sim_l \mathcal{S}$ and $\mathcal{S}$ is left-normal. The calculus is nicely simple: $\mathbb{P}_{lo}^\mu(t) = min_{\leq_L}(O_{\mathcal{R}}^\mu(t))$.

Concerning Example 5.7 we have even better possibilies. Since in a ground, orthogonal TRS, every redex occurrence is an index, *any strategy is index reduction*. We can use $\widehat{\mathbb{P}} = choose$. Clearly, for all t, and $U \subseteq O^\mu(t)$, $\nu_t(choose(U)) = choose(\nu_t(U))$ (because of non-determinism). Hence $\mathbb{P}^\mu = choose(O_{\mathcal{R}}^\mu(t))$.

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